Lecture 13 3/19

- Quasi-optimality
- Error estimation
- Lax-Milgram theory

Refs
- Johnson
- Brenner + Scott

Last time we introduced the Galerkin discretization of the Poisson problem

$$\text{(G)} \qquad (\nabla u, \nabla v) = (f, v) \qquad \text{for all } u \in V_h$$

We will now consider this as an example for a general <u>bilinear form</u> a

$$a(u, v) = (f, v)$$

$$a(\alpha u_1 + \beta u_2, \gamma v_1 + \delta v_2) = \alpha\, a(u_1, v_1)\,\gamma + \alpha\, a(u_1, v_2)\,\delta + \beta\, a(u, v_1)\,\gamma + \beta\, a(u_1, v_2)\,\delta$$

For bilinear forms, we're interested in those that generate an <u>energy norm</u> + CS

$$\|v\|_E = \sqrt{a(v, v)}, \quad a(v, w) \le \|v\|_E \|w\|_E$$

Recall from last class that this is precisely the connection between (G) and Rayleigh-Ritz

$$\text{Solving (G)} \iff \min_{v \in V_h} \|v\|_E^2$$

Last class we showed from Galerkin orthogonality

$$\|u-u_h\|_E \le \|u-v\|_E \quad \forall v \in V_h$$

or taking the inf over v

$$\|u-u_h\|_E \le \inf \|u-v\|_E \quad \forall v \in V_h$$

Energy norm vs L^2 norm

We know that (G) thus naturally minimizes error in the energy norm. What about in the L^2 norm?

$$\|u-u_h\|^2 = \int_\Omega (u-u_h)^2\,dx$$

We'll show L^2 error is <u>smaller</u> than energy error, following a <u>duality argument</u>

Define a new problem

$$-w'' = u-u_h \quad , \quad w(0)=w(1)=0$$

Then

$$\begin{aligned}\|u-u_h\|^2 &= (u-u_h, u-u_h)\\ &= (u-u_h, -w'')\\ &= ((u-u_h)', w') + \cancel{[w'(0)(u-u_h)(0) - w'(1)(u-u_h)(1)]}\end{aligned}$$

By BC

$$= a(u-u_h, w)$$

$$= a(u-u_h, w-v) \quad \text{for any } \underset{}{v}^{\in V_h}, \text{ by Galerkin orthogonality}$$

$$\leq \|u-u_h\|_E \|w-v\|_E \quad \text{by cauchy-schwartz}$$

$$\|u-u_h\| \leq \frac{\|u-u_h\|_E \|w-v\|_E}{\|u-u_h\|}$$

$$= \frac{\|u-u_h\|_E \|w-v\|_E}{\|w''\|}$$

$$\leq \|u-u_h\|_E \inf_{v \in V_h} \frac{\|w-v\|_E}{\|w''\|}$$

If we can find a ~~w~~ $v \in V_h$ such that

$$\|w-v\|_E \leq \varepsilon \|w''\| \quad (*)$$

Then

$$\|u-u_h\| \leq \varepsilon \|u-u_h\|_E$$

Applying $(*)$ again, $\|u-u_h\| \leq \varepsilon^2 \|u''\| = \varepsilon^2 \|f\|$

So everything comes down to showing that V_h can approximate things like $(*)$

For the piecewise linear choice of V_h we'll show what ε is.

Define $0 = x_0 < x_1 < \ldots < x_n = 1$ a <u>nodes</u>.

$h = \max |x_{i+1} - x$

V_h satisfying

① $v \in C^0[0,1]$

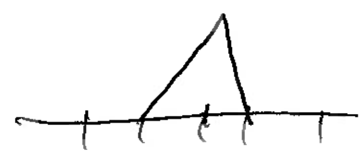
② $v\big|_{[x_{i-1}, x_i]}$ linear polynomial

③ $v(0) = 0$

Define functions $\phi_i(x)$, $i = 1, \ldots n$ as <u>nodal basis</u>

satisfying $\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Kronecker-δ property



Define the <u>interpolant</u> $\Pi u \in V_h$

satisfying $\Pi u(x_i) = u(x_i)$ $\forall$ nodes x_i

This can be computed directly

$$\Pi u(x_i) = \sum_j \widehat{\Pi u_j}\, \phi_j(x_i) = u(x_i)$$

$$= \sum_j \widehat{\Pi u_j}\, \delta_{ij}$$

$$= \widehat{\Pi u_i}$$

So $\Pi u(x_i) = \sum_j u(x_j)\, \phi_j(x)$

Finally, we'll show that Πu is the v we needed in ⊛

<u>Thm</u> $\|u - \Pi u\| \le C h \|u''\|$
$\forall u \in V$, where C is indep. of h, u

<u>Pf</u> Work on 1 element, then sum up

Show $\int_{x_{j-1}}^{x_j} (u - \Pi u)'^2 \, dx \le c (x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} u''^2 \, dx$

Let $e = u - \Pi u$ be the error, note $u'' = e''$ since Πu is linear

By change of variables

$$X = x_{j-1} + \tilde{x} (x_j - x_i)$$

We rewrite

$$\int_{x_{j-1}}^{x_j} e'^2 \, dx \le c (x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} e''^2 \, dx$$

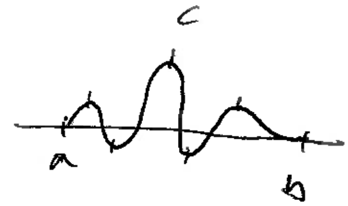
These are equivalent

as $\int_0^1 e'^2 \, d\tilde{x} \le \int_0^1 e''^2 \, d\tilde{x}$

To prove this is a little bit of calculus

We'll use Rolle's theorem, which is the mean value theorem in the special case where the endpoints are equal

Thm if e is continuous on $[a, b]$ and $f(a) = f(b)$, then there is at least one pt $c \in [a, b]$ s.t $f'(c) = 0$



so $$e'(\xi) = 0 \quad \text{for some } \xi \in [0, 1]$$

$$e'(y) - \underbrace{e'(\xi)}_{0} = \int_{\xi}^{y} e''\, dx$$ Fund Thm of Calc.

$$|e'(y)|^2 = \left| \int_{\xi}^{y} e''\, dx \right|^2$$

$$= \left| \int_{\xi}^{y} 1 \cdot e''\, dx \right|^2$$

$$\leq \left| \int_{\xi}^{y} dx \right| \left| \int_{\xi}^{y} e''\, dx \right|$$

$$= |y - \xi| \left| \int_{\xi}^{y} e''\, dx \right|$$

$$|e'(y)| \leq |y - \xi|^{1/2} \left| \int_{0}^{1} e''\, dx \right|^{1/2}$$

$$\int e'(y)^2\,dy \le \int_0^1 |y-\xi|\,dy \int_0^1 e''\,dx$$

Taking max over ξ gives worst case scenario when $\xi = 1/2$

$$\max \int_0^1 |y-\xi|\,dy = \frac{1}{2}$$

And we're done

We just showed (A) holds w/ $\varepsilon = h^2$

So: $\|u-u_h\| \le ch^2\|f\|$

Summarize

(1) $\|u-u_h\|_E \le \|u-v\|_E \quad \forall v \in V_h$

Picking $v = \Pi u \in V_h$

$\|u-u_h\|_E \le c_1 h \|\underbrace{u''}_{=f}\| \le c_1 h\|f\|$

$\|u-u_h\| \le c_2 h^2 \|\underbrace{u''}_{=f}\| \le c_2 h^2 \|f\|$

What's up next:

- Machine learning V_h
- Machine learning $a(u,v)$

First, an abstraction of what we learned today, so that its clear this isn't just something special about Poisson

Lax - Milgram theory

There isn't anything special about poisson - the same process holds for any elliptic billinear forms

Let V be a Hilbert space w/ norm $\|\cdot\|_V$

Suppose $a(u,v)$ satisfies

1. Symmetric
$$a(u,v) = a(v,u)$$

2. Continuous

Have a Cauchy-Schwartz

$\exists\ \gamma > 0$ s.t

$\forall v, w \in V$
$$|a(v,w)| \leq \gamma \|v\|_V \|w\|_V$$

3. Elliptic (aka V-elliptic, coercive)
$$a(v,v) \geq \alpha \|v\|_V^2 \qquad \forall v \in V$$

Then
$$F(u) = \min_v F(v) \quad \& \quad a(u,v) = L(v)$$
$$F(v) = \tfrac{1}{2} a(v,v) - L(v)$$

for
$$|L(v)| \leq \Lambda \|v\|_V$$

have the same unique solution satisfying $\|u\|_V \leq \dfrac{\Lambda}{\alpha}$

Pf $\quad \min F(v) \Rightarrow a(u,v) = L(v)$ $\quad$ immediate from $\delta_v F(v) = 0$

To obtain estimate take $v = u$

$$a(u,u) = L(u)$$

$$\alpha \|u\|_V^2 \leq \qquad \leq \Lambda \|u\|_V$$

$$\|u\|_V \leq \frac{\Lambda}{\alpha}$$

For uniqueness assume $\exists$ solns u_1, u_2

$$a(u_1 - u_2, v) = 0$$

From stability result

$$\|u_1 - u_2\|_V \leq \frac{1}{\alpha} \|RHS\| = 0$$

For <u>any</u> elliptic operator, we have the playbook

- Get quasi-optimality for best fit in $\|\cdot\|_V$
- Get interpolant as upper bound
- Relate $\|\cdot\|_V$ to $\|\cdot\|$ to understand how quickly error converges