Last time

- April 30th Last Day
- May 5-13 finals
- Plan projects now

Lax-Milgram - abstract stability analysis

(G) Find $u \in V_h$ s.t. $V_h \subseteq V$

$$a(u,v) = L(v) \quad \forall v \in V_h$$

ex Poisson

$$-\nabla^2 u = f$$

$$u|_{\partial\Omega} = 0$$

$$a(u,v) = \int \nabla u \cdot \nabla v \, dx, \quad L(v) = \int f v \, dx$$

Biharmonic Eqn (Acoustics, phase field, Beam theory)

$$\rightarrow \quad \nabla^2 \nabla^2 u = f$$

$$u|_{\partial\Omega} = \partial_n u|_{\partial\Omega} = 0$$

$$V = H^2(\Omega) = \{ \int (\nabla^2 u)^2 < \infty \}$$

$$a(u,v) = \int \nabla^2 u \cdot \nabla^2 v \, dx \qquad L(v) = \int f v \, dx$$

Lax-Milgram

a is cont ① $|a(v,w)| \leq \gamma \|v\|_V \|w\|_V$

a is elliptic ② $a(v,v) \geq \alpha \|v\|_V^2$ $\Rightarrow$ Stability

L is cont ③ $|L(v)| \leq \Lambda \|v\|_V$

Cea's Lemma

Suppose ① - ③. Then

$$\|u-u_h\|_V \le \frac{\gamma}{\alpha} \inf_{v \in V_h} \|u-v\|_V$$

<u>pf</u> $u-u_h \in V$

$a(u-u_h, v) = 0 \quad \forall v \in V_h$

$$\begin{aligned}
\alpha \|u-u_h\|_V^2 &\le a(u-u_h, u-u_h) \\
&= a(u-u_h, u) - a(u-u_h, u_h) \\
&= a(u-u_h, u) - a(u-u_h, v) \quad \text{for any } v \\
&= a(u-u_h, u-v) \\
&\le \gamma \|u-u_h\|_V \|u-v\|_V
\end{aligned}$$

$$\|u-u_h\|_V \le \frac{\gamma}{\alpha} \|u-v\|_V \quad \forall v \in V_h$$

Today we'll get some practice working through this abstract theory

We are free to choose the

- PDE
- FEM subspace
- Norm

Remember

$L^2 \to \|f\|_{L_2} = \int f^2 dx$

H^1-seminorm

$$|f|_{H_1} = \int \nabla f^2 dx$$

H^1-norm

$$\|f\|_{H_1}^2 = \|f\|_{L_2}^2 + |f|_{H_1}^2$$

ex

$$a(u,v) = \int \nabla u \cdot \nabla v \, dx$$

$$L(v) = \int f v \, dx$$

Let $V = H_0^2(\Omega)$ ← ~~twice differentiable~~ integrable 2^{nd} derivs

$$(u,v)_V = \int D^2 u : D^2 v + \nabla u \cdot \nabla v + uv$$

where $A:B = \sum_{ij} A_{ij} B_{ij}$ Frobenius inner prod.

Check conditions

① $|a(v,w)| = \left| \int \nabla v \cdot \nabla w \, dx \right|$

$$\leq \|\nabla v\|^* \|\nabla w\|^* \quad \text{C-S}$$

$\|f\| = \int f^2 dx$

$$\|v\|_V^2 = \int D^2 v : D^2 v + |\nabla v|^2 + v^2 \, dx$$

$$\geq \int |\nabla v|^2 dx = \|\nabla v\|$$

so $|a(v,w)| \leq \|v\|_V \|w\|_V$

②

$$a(v,v) = \|\nabla v\| \geq \alpha \int D^2 v + |\nabla u|^2 + v^2 \, dx$$

~~Challeng.~~ <u>Non-Trivial</u> $\rightarrow$ need a result from PDEs called elliptic regularity

For convex Ω, $\|v\|_{H^2(\Omega)} \leq C \|\Delta v\|_{L^2(\Omega)}$

for $v \in H^2 \cap H_0^1$

③ $|L(v)| \leq \|f\|_{L^2} \|v\|_{L^2}$

$\leq \|f\|_{L^2} \|v\|_V \Rightarrow \Lambda = \|f\|_{L^2}$

Reaction Diffusion eqn

$$-\nabla \cdot A \nabla u + r u = f$$

w/assumptions (A) A is pos def, $x^T A x > 0$

(B) $r \geq 0$

(C) $|A_{ij}| < \infty$, $|r| < \infty$ bounded

weak form

$a(u,v) = L(v) \qquad v \in H_0^1$

$a(u,v) = \int_\Omega \nabla u \cdot A \cdot \nabla v + r u v \, dx$

$L(v) = \int_\Omega f v \, dx$

(1) show $|a(v,w)| \leq \gamma \|v\|_V \|w\|_V$

$$\left| \int \nabla v \cdot A \cdot \nabla w \, dx \right| \leq \max_{i,j} |A_{ij}| \left| \int \nabla v \cdot \nabla w \, dx \right|$$

$$\cancel{\leq \gamma_A \|v\|_{H_1} \|w\|_{H_1}}$$

$$\leq \gamma_A |v|_{H_1} |w|_{H_1}$$

$$\left| \int r v w \, dx \right| \leq \max |r| \int v w \, dx$$

$$\leq \gamma_r \|v\|_{L^2} \|w\|_{L^2}$$

$$a(v,w) \leq \gamma_A |v|_{H_1} |w|_{H_1} + \gamma_r \|v\|_{L^2} \|w\|_{L^2}$$

$$\leq 2\max(\gamma_A, \gamma_r) \|v\|_{H_1} \|w\|_{H_1}$$

(2) $a(u,u) \geq \alpha \|u\|_V^2$

$$a(u,u) = \int \nabla u \cdot A \cdot \nabla u \;\cdot\; r u v \; dx$$

$$\geq \int \nabla u \cdot \nabla u + u^2 \; dx \qquad \text{by (A)+(B)}$$

$$= \|u\|_{H_1}^2$$

Elasticity

- First example of how analysis predicts breakdown of stability

<u>def</u> $u \in \mathbb{R}^d$ – displacement

$\varepsilon(u) \in \mathbb{R}^{d \times d}$ – strain

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

$\sigma(\varepsilon)$ – stress

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$$

To enforce rotational invariance for isotropic materials

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

so the stress reduces to

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$$

<u>Aside</u> For elasticity, we have the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} \varepsilon^T C \varepsilon + \frac{\rho}{2} \partial_t u^2$$

Governing Eqns

$$\begin{cases} -\operatorname{div} \sigma = f \\ \sigma = 2\mu\varepsilon + \lambda\, \mathrm{tr}(\varepsilon) I \\ u|_{\partial\Omega} = 0 \end{cases}$$

Note that $tr(\varepsilon) = \nabla\cdot u$

$$\begin{cases} -\nabla\cdot\left(2\mu\,\varepsilon(u) + \lambda\,\nabla\cdot u\, I\right) = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

To define variational problem, we'll work w/ displacements

$\vec{u} \in \vec{H}_0^1(\Omega) = \left[H_0^1(\Omega)\right]^d$ aka $\leftarrow$ H_0^1 in each component (lots of other choices)

(V) $\Rightarrow$ $a(u,v) = (f,v)$

$$a(u,v) = \int \mu\,\varepsilon(u):\varepsilon(v) + \lambda(\nabla\cdot u)(\nabla\cdot v)\,dx$$

Again, we'll check continuity and ellipticity of a,

but write $\lambda = \dfrac{2\mu\,\delta}{1-4\delta}$, where δ is the Poisson ratio

$$\delta = -\frac{\varepsilon_{transverse}}{\varepsilon_{axial}}$$

For near-incompressible materials, as $\delta \to \frac{1}{2}$, $\lambda \to \infty$

So we'd like to understand the $\lambda \to \infty$ limiting stability

ex near-incomp

Rubbers

Hydrogels

Biomaterial (cartilage, brain)

Wet clays (saturated porous)

To do this we need Korn's inequality

$$\int \varepsilon(u):\varepsilon(u)\,dx \geq \|u\|_{H_1}$$

① $|a(w,v)| \leq \gamma \|w\|_{H_1} \|v\|_{H_1}$

$$\left|\int \mu\, \varepsilon(w) : \varepsilon(v)\, dx\right| = \left|\frac{\mu}{4} \sum_{i,j} \int (\partial_{x_i} w_j + \partial_{x_j} w_i) \cdot (\partial_{x_i} v_j + \partial_{x_j} v_i)\right|$$

$$= \left|\mu \sum_{i,j} \int \partial_{x_i} w_j \; \partial_{x_i} v_j \, dx\right|$$

$$\cancel{= \mu \|u\|_{H_1}}$$

$$\leq \mu \|w\|_{H_1} \|v\|_{H_1}$$

$$\left|\int \lambda (\nabla \cdot w) \nabla \cdot v\right|$$

$$\leq \lambda \int |\nabla \cdot w| \int \nabla \cdot v$$

$$\leq \lambda \|w\|_{H_1} \|v\|_{H_1}$$

$$|a(w,v)| \leq (\mu + \lambda) \|w\|_{H_1} \|v\|_{H_1}$$

② $a(u,u) = \int \mu\, \varepsilon(u) : \varepsilon(u) + \lambda (\nabla \cdot u)^2 \, dx \overset{?}{\geq} \alpha \|u\|_{H_1}^2$

$$\geq \int \mu\, \varepsilon(u) : \varepsilon(u)\, dx$$

$$\geq \mu \|u\|_{H_1} \quad \text{By Korn}$$

what goes wrong?

① $\|u_h\|_{H_1} \leq \frac{\Lambda}{\mu}$ $\Rightarrow$ get a stable solution

loss of control!

② By Cea's lemma $\|u - u_h\|_{H_1} \leq \frac{\mu + \lambda}{\mu} \|u - v\|_{H_1}$

One solution, and our first example of <u>mixed FEM</u>

Take

$$\mu \int \varepsilon(u):\varepsilon(v) + \lambda \int \operatorname{div} u \operatorname{div} v = \int f v$$

Let $p = \lambda \operatorname{div} u$, and introduce <u>second</u> FEM space $q \in M_h$

$$\mu \int \varepsilon(u):\varepsilon(v)\,dx + \int p \nabla\cdot v\,dx = \int f v\,dx$$

$$\int \nabla\cdot u\, q\,dx + \frac{1}{\lambda}\int p q\,dx = 0$$

This gives a <u>new galerkin eqn</u> but how to design M_h?

<u>Note</u> in limit $\mu = 1$, $\lambda \to \infty$ this reduces to the <u>stationary stokes problem</u>

(S) $$\begin{cases} -\Delta u - \nabla p = f \\ \nabla\cdot u = 0 \end{cases}$$

$u \in H_0^1$ for Laplacian to make sense, $M_h \subseteq L^2$

$$(\nabla p, v) \xrightarrow[\text{IBP}]{} -(p, \nabla\cdot v) < \|p\|_{L^2}\|v\|_{H_1}$$

Galerkin form

$$(\nabla u, \nabla v) + (p, \nabla\cdot v) = (f, v)$$
$$(\nabla\cdot u, q) = 0$$

~~Stability~~ ~~Take~~ $\cancel{q=p,\ v=u}$

$$\cancel{\|u\|_{H_1}^2 + (p, \nabla u)} = (f, u)$$

This gives our first variational saddle-pt problem. We'll need some lemmas to tackle it, but lets prove uniqueness quickly now ($f=0 \Rightarrow p,u=0$)

Take $v=u$

$$\|u\|_{H_1}^2 + (p, \nabla\cdot u) = 0$$

Taking $q=p$ in the 2^{nd} eqn, $(p, \nabla\cdot u) = 0 \Rightarrow \|u\|_{H_1}^2 = 0$

$$\Rightarrow u = 0$$

Finally, we'd like to show $u=0 \Rightarrow p=0$. But we can't. Nothing in eqn gives us that

IDEA Design relationship between V_h, M_h so $u=0 \Rightarrow p=0$

Inf-sup condition

V_h and M_h are inf-sup compatible if for any $q \in M_h$, there exists a $v \in V_h$

s.t
$$\beta \|q\|_{L^2} \leq \frac{(q, \nabla \cdot v)}{\|v\|_{H^1}}$$

Returning to Stokes momentum eqn

$$(\nabla u, \nabla v) + (p, \nabla \cdot v) = 0$$

Already shown

Take $q = p$

$$\|p\|_{L^2} \leq \frac{(p, \nabla \cdot v)}{\beta \|v\|_{H_1}} = 0$$

$$\Rightarrow p = 0$$

Next time

- How to build inf-sup stable spaces
- How to learn physics in this mixed FEM setting