Last time we introduced <u>mixed FEM</u> spaces for Stokes flow

$$\nabla^2 u - \nabla p = f$$
$$\nabla \cdot u = 0$$

This is an example of the general class of saddle-point problems

$$a(u,v) + b(p,v) = L_1(v) \qquad \forall (v,q)$$
$$b(q,u) = L_2(q) \qquad \in V_h \otimes M_h$$

Assuming a satisfies Lax-Milgram conditions, lets repeat the uniqueness proof from last lecture

Let $L_1, L_2 = 0$, $q = p$, $v = u$

$$a(u,u) + b(p,u) = 0$$
$$b(p,u) = 0$$

$$\alpha \|u\|_V \leq a(u,u) = 0$$
$$\Rightarrow u = 0$$

Eqn 1 reduces to

$$b(p,v) = 0$$

If V_h, M_h satisfy inf-sup compatibility

i.e for all $p \in M_h$, $\exists\, \tilde{v} \in V_h$ s.t

$$\frac{b(p,\tilde{v})}{\|\tilde{v}\|_V} \geq \beta \|p\|_M$$

Then
$$\|p\|_M \leq 0$$
$$p = 0$$

Back to ML

So far we've learned models like

$$Au + \varepsilon N(u) = f$$

Now we have the equipment to consider conservation laws

$$\nabla \cdot F = f$$

$$F = \underset{\text{linear}}{L(u)} + \varepsilon \underset{\text{nonlinear}}{N(u)}$$

These satisfy, for $f = 0$

$$0 = \int_\Omega f\,dx = \int_\Omega \nabla \cdot F\,dx = \int_{\partial\Omega} F \cdot dA$$

Similarly, one can define ~~similarly~~

$$\nabla \times F = f$$

$$F = L(u) + \varepsilon N(u)$$

For conservation of circulations (e.g. magnetic fields, vorticity ...

$$0 = \int f\,\cancel{dA} = \int \nabla \times F \cdot dA = \oint F \cdot dl$$

Lets start by designing a good space in the linear setting ($\varepsilon = 0$) in 1D

$$F' = f$$

$$F + L(u)$$

The first equation gives

IBP

$$(q, F') = - (q', F) = (f, q)$$

$$b(q, F) = L_2(q)$$

To obtain a saddle point problem (and an automatic stability, can choose

$$L(u) = u'$$

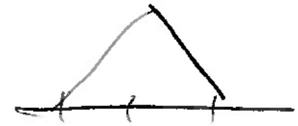
$$(F, v) + (u', v) = 0$$

$$a(F, v) + b(u, v) = 0$$

w/ $\quad a(u, v) = (u, v)$

- Can easily check a satisfies Lax-Milgram
- Just need to choose an inf-sup stable V_h, M_h

Let $M_h = \{ f \in L^2(\Omega) ,\ f|_e = \text{const} \}$



V_h = piecewise linears $\subseteq H_0^1$

Given a p, we can build $\tilde{v}$ such that $\tilde{v}' = p$

Set $\tilde{v}_0 = 0$

$$\tilde{v}_j = \sum_{k=0}^{j-1} p_k h$$

Then $\int_{e_j} \nabla \tilde{v} \, dx = \tilde{v}_{j+1} - \tilde{v}_{j} = p_j h$

Note that $b(p, \tilde{v}) = \int p \tilde{v}'$

$$= \sum_j \hat{p}_j \int \chi_{[x_j, x_{j+1}]} \tilde{v}'$$

$$= \sum_j \hat{p}_j \left(\tilde{v}_{j+1} - \tilde{v}_j \right)$$

$$= \sum_j \hat{p}_j^2 h$$

$$= \|p\|_{L^2}^2$$

And $$|v|_{H_1}^2 = \sum_{e_j} \int_{x_j}^{x_{j+1}} \nabla v^2 \, dx$$

$$= \sum_{e_j} \int_{x_j}^{x_{j+1}} p^2 \, dx$$

$$= \| p \|_{L^2}^2$$

So $$\frac{b(p, \tilde{v})}{|v|_{H_1}} = \frac{\|p\|_{L^2}^2}{\|p\|_{L^2}} = \|p\|_{L^2}$$

Now that we've worked a pair of inf-sup stable spaces out, notice

① that for any bigger choice of $V_h \subseteq V_h^{big}$, $\tilde{v} \in V_h^{big}$, and so M_h and V_h^{big} are also inf-sup compatible

② The key property was that V_h is chosen big enough that $\tilde{v}' = p$

or the derivative operator is onto (surjective)

$$\frac{d}{dx} V_h \supseteq M_h$$

Thm
In higher dimensions, for $D = \mathrm{div}/\mathrm{grad}/\mathrm{curl}$
$D^* = \mathrm{grad}/\mathrm{div}/\mathrm{curl}$

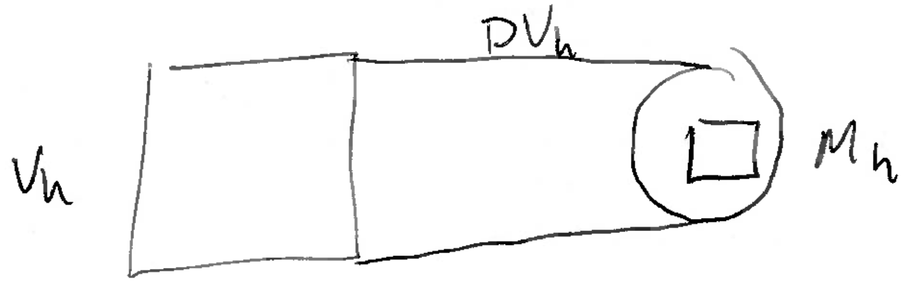
$$(D^* f, g) = (f, Dg)$$

The saddle point problem

$$(F, v) - (Du, v) = 0$$

$$(q, D^* F) = (f, q)$$

is inf-sup stable if $DV_h \supseteq M_h$



Now we have FEM spaces nailed down
What about discrete conservation properties?

Consider higher-dim now

$$-(\nabla q, F) + \langle q, F \rangle_{BC} = (f, q)$$

Global conservation

Let $g = 1$

$$-(\cancelto{0}{\nabla g}, F) + \langle g, F\rangle = (f, g)$$

$$\int_{\partial\Omega} F \cdot dA = \int_{\Omega} f \, dx$$

flux in/out $=$ source - sinks

Local conservation.

Let ~~ϕ_i~~ $V_h = \text{span}(\phi_i)$

Let $g = \phi_i$

Assume $F \cdot dA = 0$

$$\sum_i \phi_i = 1$$

$$-\int \nabla\phi_i \cdot F \, dx = (f, \phi_i)$$

$$= -\sum_j \int \phi_j \nabla\phi_i \cdot F \, dx$$

$$\sum_j \nabla\phi_j = 0$$

$$\sum_j \phi_i \nabla\phi_j = 0$$

$$= \sum \int \underbrace{(\phi_j \nabla\phi_i - \phi_i \nabla\phi_j)}_{\psi_{ij}^1 = -\psi_{ji}^1} \cdot F \, dx$$

If $f = 0$

$$\sum_j \underbrace{\int \psi_{ij}^1 \cdot F \, dx}_{\text{equal and opposite \underline{local} fluxes}} = 0$$

Denote $W^0 = \text{span}(\phi_i)$

$W^1 = \text{span}(\psi^1_{ij})$

<u>Thm</u> $\text{grad}(W^0) \subseteq W^1$

<u>pf</u> Let $f \in W^0$

$$f = \sum \hat{f}_i \, \phi_i(x)$$

$$\nabla f = \sum_i \hat{f}_i \, \nabla\phi_i$$

$$= \sum_{i,j} \hat{f}_i \left(\phi_j \nabla\phi_i - \phi_i \nabla\phi_j\right)$$

$$= \sum_{i,j} \hat{f}_i \, \psi^1_{ij} \subseteq W^1$$

Motivated by the construction, we'll "play by ear" and make a space for curls

$$(\psi^1_{ij}, \nabla\times F) = (\nabla\times\psi^1_{ij}, F)$$

$$= \int \nabla\times(\phi_j\nabla\phi_i - \phi_i\nabla\phi_j) \cdot F \, dx$$

$\nabla\times f\vec{G}$

$= \nabla f\times G +$

$f\,\nabla\times G$

$$= \int 2\,\nabla\phi_j \times \nabla\phi_i \cdot F \quad dx$$

$$= \sum_k 2\,\phi_k \,\nabla\phi_j + \nabla\phi_i \cdot F \, dx$$

$$= \sum_K 2 \underbrace{\left(\phi_K \nabla\phi_j \times \nabla\phi_i + \phi_i \nabla\phi_K \times \nabla\phi_j + \phi_j \nabla\phi_i \times \nabla\phi_K \right)}_{\psi^2_{ijK}} \cdot F$$

← anti-symmetric wrt to index swap.

Let $W^2 = \mathrm{span}(\psi^2_{ijK})$

<u>Thm</u> $\mathrm{curl}(W^1) \subseteq W^2$

Can now build a <u>deRham complex</u>

$$W^0 \xrightarrow{\text{grad}} W^1 \xrightarrow{\text{curl}} W^2$$

Preserving exactly $\mathrm{curl} \circ \mathrm{grad} = 0$

& $$W^0 \xleftarrow[-\text{div}]{} W^1 \xleftarrow[\text{curl}]{} W^2$$

integrating by parts $(\nabla f, g) = -(f, \nabla \cdot g)$ gives

$\mathrm{div} \circ \mathrm{curl} = 0$

Return to nonlinearity

$$Lu + \varepsilon N(u) = f$$

$$a(u,v) + \varepsilon (N(u),v) = (f,v)$$

We introduce a new condition

<u>Monotonicity</u> $\quad (N(u),u) \geq \alpha_N \|u\|_V^2$

To prove stability

$$\alpha \|u\|_V^2 - \varepsilon \alpha_N \|u\|_V^2 \leq a(u,u) + \varepsilon (N(u),u) = (f,u) \leq \Lambda \|u\|_V$$

$$\|u\|_V \leq \frac{\Lambda}{\alpha - \varepsilon \alpha_N} \|f\|_V$$