Some vector calc review

"Div/Grad/Curl and all that" Schey

Recall the Levi-Civita tensor ε_{ijk}

~~tmp~~ satisfying
$$\varepsilon_{ijk} = \begin{cases} 1 & \text{even permutations of } (i,j,k) \\ -1 & \text{odd} \quad " \\ 0 & \text{equal indices} \end{cases}$$

Provides a compact notation for curl

$$\vec{F} \times \vec{G} = \varepsilon_{ijk} F_j G_k$$

Some identities - aka exact sequence property

$$\nabla \cdot \nabla \times u = \nabla \times \nabla \phi = 0$$

Adopting Einstein notation

$$A_{ii} = \sum_i A_{ii}$$

Pf

$$\nabla \times u = \varepsilon_{ijk} \partial_{x_j} u_k$$

$$\begin{aligned} \nabla \cdot \nabla \times u &= \varepsilon_{ijk} \partial_{x_i} \partial_{x_j} u_k \\ &= \frac{1}{2} \varepsilon_{ijk} \partial_{x_i} \partial_{x_j} u_k + \frac{1}{2} \varepsilon_{jik} \partial_{x_j} \partial_{x_i} u_k \\ &= \frac{1}{2} \left(\varepsilon_{ijk} + \varepsilon_{jik} \right) \partial_{x_i} \partial_{x_j} u_k \\ &= 0 \end{aligned}$$

odd permutation

Similarly,

$$\nabla\times\nabla\phi = \varepsilon_{ijk}\,\partial_{x_j}\partial_{x_k}\phi$$

$$= \frac{1}{2}\left(\varepsilon_{ijk} + \varepsilon_{ikj}\right)\partial_{x_j}\partial_{x_k}\phi$$

$$= 0$$

Thm Helmholtz-Hodge decomposition

Given a smooth vector field $F \in \mathbb{R}^d$ on a simply connected subset of $\mathbb{R}^d$

there exist unique ϕ, A s.t. $F = \underbrace{-\nabla\phi}_{\substack{\text{scalar potential}\\ \text{curl-free}}} + \underbrace{\nabla\times A}_{\substack{\text{vector potential}\\ \text{div-free}}}$

Pf

$$\nabla\cdot F = -\nabla\cdot\nabla\phi + \cancel{\nabla\cdot\nabla\times A}$$

Solve $\begin{cases} -\nabla^2\phi = \nabla\cdot F \\ \partial_n\phi|_{\partial\Omega} = -F\cdot\hat{n} \end{cases}$ aka pure neumann Poisson problem

Note that this problem has a constant vector in its null-space

i.e. for any c
$$\nabla^2(\phi+c) = \nabla^2\phi$$
$$\partial_n(\phi+c) = \partial_n\phi$$

To maintain coercivity in Lax-Milgram, need to work in space orthogonal to kernel

$$V_h = \{\text{CPWL functions } u \mid u\cdot\vec{1} = 0\}$$

e.g. $\min_{\phi} \int_{\Omega} \frac{1}{2} |\nabla\phi|^2 - \phi \nabla\cdot F + \lambda(\phi \cdot \vec{1})$

$$\stackrel{\phi}{-} \int_{\partial\Omega} F\phi \cdot dA$$

Would be a way to enforce w/ Lagrange multiplier

- That gave ϕ, now to get A

$$\nabla\times F = -\cancel{\nabla\times\nabla\phi} + \nabla\times\nabla\times A$$

Now we have a more challenging null space

For any q, $\nabla\times\nabla\times(A + \nabla q) = \nabla\times\nabla\times A$

So we'd like solutions orthogonal to grad q, for any q

$$(A, \nabla q) = 0 \quad \forall\, q$$

Applying Galerkin

$$(\nabla\times\nabla\times A, v) = (\nabla\times F, v)$$

$$= (\nabla\times A, \nabla\times v) + \underbrace{\int_{\partial\Omega} (\nabla\times u \times \hat{n})\cdot v \, dA}_{\text{Natural BC}} = (\nabla\times F, v)$$

$\nabla\times u \times \hat{n} = \nabla\times F \times \hat{n}$

We can enforce w/ Lagrange mult. again

$$(\nabla\times\nabla\times A, v) + (\nabla\lambda, v) = (\nabla\times F, v) - \langle \nabla\times F\times\hat{n}, v\rangle$$

$$(A, \nabla q) = 0$$

It'd be more work to show, but this is a saddle point problem that we showed how to approach last week.

$\blacksquare$

The Hodge decomposition is a useful tool for projecting onto a div-free or curl-free space.

e.g

$$\Pi_{div\text{-}free}\, u = \nabla\times A$$

$$\Pi_{curl\text{-}free}\, u = -\nabla\phi$$

Ex fluid Mechanics

Potential flow was an early model of fluids (w/ Large, restrictive assumptions) that was used e.g. to design air foils in 20's - 40's

Gives lift, not drag

Assume
$$\begin{cases} u = -\nabla \phi \\ \nabla \cdot u = 0 \end{cases}$$

Then
$$\begin{cases} -\nabla^2 \phi = 0 \\ \partial_n \phi |_{\partial\Omega} = u \cdot \hat{n} \end{cases}$$

No FEM back then → Laplace transforms etc.

You'll still get this in a graduate fluids course

Ex Fluid Mechanics

When solving Navier-Stokes w/ incompressibility

$$\partial_t u + u\cdot\nabla u = -\nabla p + \nu\nabla^2 u$$
$$\nabla\cdot u = 0$$

The incompressibility constraint is annoying
- Makes system bigger
- How to choose inf-sup u, p?
- Couples all points in domain

A popular scheme (Alexandre Chorin, 1968)

Called a splitting-scheme

① $$\frac{u^\star - u^n}{K} = -u^n\cdot\nabla u^n + \nu\nabla^2 u^\star$$

②a $$\frac{u^{n+1} - u^\star}{K} = -\nabla p^{n+1}$$

②b $$\nabla\cdot u^{n+1} = 0$$

Taking div of ②a

$$\cancel{\nabla\cdot u^{n+1}} - \frac{\nabla\cdot u^\star}{K} = -\nabla^2 p^{n+1}$$

or $$\boxed{u^{n+1} - u^\star = \underset{\text{divfree}}{\Pi} u^\star - u^\star}$$ pressure-projection

Back to graphs

We introduced grad/div as coboundary, codifferential pair

$$\langle v, \delta_0 u \rangle = \langle \delta_0^* v, u \rangle$$

Now for the formal def. so we can talk curls.

<u>def</u> k-chain $C^k = [v_1 \; \ldots \; v_k]_{v_i \in V}$ anti-symmetric w.r.t. swaps

<u>def</u> boundary $\partial_{k-1} : C^k \to C^{k-1}$

$$\partial_{k-1}[v_1, \ldots v_k] = \sum_{i=1}^{k} (-1)^{i-1} [n_1, \ldots \hat{n}_i, \ldots n_k]$$

means leave this vertex out

<u>ex</u> For edge $[v_1, v_2] = -[v_2, v_1]$

$$\partial_0 [v_1, v_2] = (-1)^0 [v_2] + (-1)^1 [v_1] = v_2 - v_1$$

Thm Exact sequence property

$$\partial_{k-1} \circ \partial_k = 0$$

pf/ Given $c \in C^{k+1}$

$$\partial_{k-1}\partial_k c = \sum_{i=1}^{k-1} (-1)^{i-1} \partial_k [n_1, \dots \hat{n}_i, \dots n_{k-1}]$$

$$= \sum_{i=1} \sum_{j} (-1)^{i-1} (-1)^{j-1} [n_1, \dots \hat{n}_i, \dots \hat{n}_j \dots n_k]$$

anti-symmetric if we exchange i, j

$$= \sum_{i,j} (-1)^{i+j-2} \alpha_{ij}$$

$\alpha_{ij} = -\alpha_{ji}$

$$= 0$$

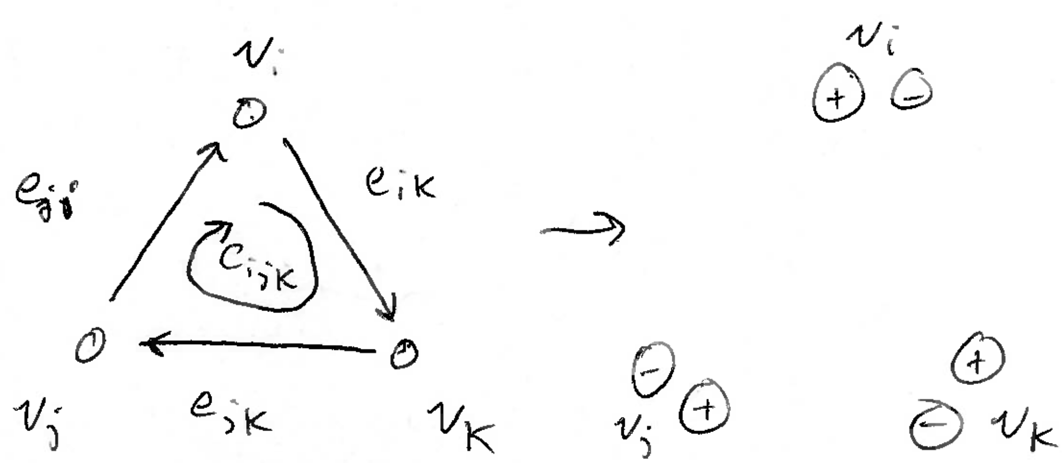
∎

Illustrative sketch

$c_{ijk} = [v_i, v_j, v_k]$

$\partial c_{ijk} = e_{ji} + e_{ik} + e_{jk}$

$= [v_j, v_i] + [v_i, v_k] + [v_k, v_j]$



$\partial\partial c_{ijk} = v_j - v_i + v_i - v_k + v_k - v_j$

To define the coboundary (div/grad/curl) just swap chains w/ cochains

For $f \in C_K$

~~$d_{K-1} f(v_1, \dots v_K) = \sum (-1)^i f($~~

$$\delta_{K-1} f = \sum (-1)^i f(v_1, \dots \hat{v}_i, \dots v_K)$$

ex

$$\delta_0 \phi_{ij} = \phi_j - \phi_i \qquad \text{graph gradient}$$

$$\delta_1 A_{ijK} = A_{ij} + A_{jK} + A_{Ki} \qquad \text{graph curl}$$

$$\text{curl} \circ \text{grad} = 0$$

$$\begin{aligned} \delta_1 \delta_0 \phi &= \delta_0 \phi_{ij} + \delta_0 \phi_{jK} + \delta_0 \phi_{Ki} \\ &= \phi_j - \phi_i + \phi_K - \phi_j + \phi_i - \phi_K \\ &= 0 \end{aligned}$$

If $\quad \delta_1 \delta_0 = 0, \quad \delta_0^* \delta_1^* = 0 \qquad$ (div of curl $= 0$)

$\underline{\text{pf}}$ $\quad \langle f, \delta_0^* \delta_1^* g \rangle$

$$= \langle \delta_0 f, \delta_1^* g \rangle$$
$$= \langle \delta_1 \delta_0 f, g \rangle$$
$$= 0$$

$$\boxed{\begin{tikzcd} C_1 \arrow[r, shift left, "\delta_0"] & C_2 \arrow[l, shift left, "\delta_0^*"] \arrow[r, shift left, "\delta_1"] & C_3 \arrow[l, shift left, "\delta_1^*"] \end{tikzcd}}$$

Recall

$$\delta_0 = \text{grad} \qquad \delta_0^* = \text{div}$$
$$\delta_1 = \text{curl} \qquad \delta_1^* = \text{curl}$$

Graph Hodge Decomposition

$\underline{\text{thm}}$ Given $\quad f \in C_K$

$$f = \delta_0 \phi + \delta_1^* A$$

$\underline{\text{pf}}$

$$\delta_0^* f = \delta_0^* \delta_0 \phi + \cancel{\delta_0^* \delta_1^* A}$$
$$\delta_1 f = \cancel{\delta_1 \delta_0 \phi} + \delta_1 \delta_1^* A$$

Given observations on some preferences Y^{data}_{ij}

$$\min_{S} \sum_{ij} w_{ij} \left(Y^{data}_{ij} - \delta s_{ij} \right)^2$$

Take the variation of this (HW exercise)

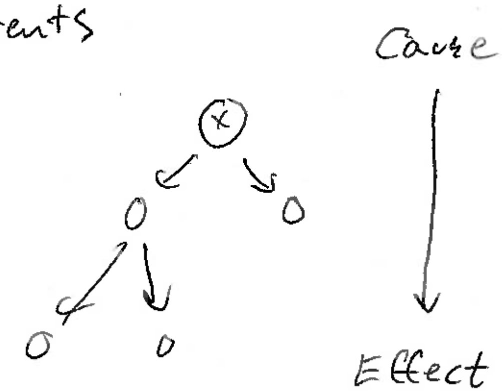
$$\boxed{\delta_0^{*} W \delta_0 S = \delta_0^{*} W Y^{data}}$$

see papers on git, 1

Causality

Given a set of events X_i, a causal structural model is a decomposition

$$p(x_1 \dots x_N) = \prod_{i=1}^{N} P(X_i \mid \underbrace{Pa(x_i)}_{\text{parents}})$$



Define a score S_i

flux $F = \delta_0 S$

and parent $Pa(x_i) = \{ x_j \text{ s.t. } F_{ij} \geq 0 \}$

or a differentiable version

$$Pa(x_i) = \left\{ ReLU\left(\tanh\left(\tfrac{1}{\beta} F_{ij} \right) \right) > 0 \right\}$$