Ref Gustafsson et al ch 1-2

Today

- Pytorch, Linear Model Eqns, Analysis essentials

- Consider the domain $\Omega = [0, 2\pi]$ periodic

For PDEs that are 2nd order and linear

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F = 0$$

The PDE is
- parabolic if $B^2 - AC = 0$
- hyperbolic $B^2 - AC > 0$
- elliptic $B^2 - AC < 0$

- For $A, B \ldots F$ varying w/space/time, the behavior could be mixed
- For Non-linear PDEs, linearization will show localized behavior

Parabolic - One real characteristic direction
- ex Heat/diffusion eqn

Hyperbolic - Two distinct real characteristics
ex wave equation.

Elliptic - No real characteristics
ex Laplace / steady-state diffusion

$\boxed{\text{Different Methods for Each!}}$

For each of these we will make use of the following analytic solutions w/ periodic BC $u_K(0) = u_K(2\pi)$

- Transport Eqn

$$\partial_t u_K + \partial_x u_K = 0$$

$$u_K = \sin\left(2\pi K(x-t)\right)$$

- Unsteady heat eqn

$$\partial_t u_K + \partial_{xx} u_K = 0$$

$$u_K = \exp\left(4\pi^2 K^2 t\right) \sin\left(2\pi K x\right)$$

- Forced Poisson eqn

$$\partial_{xx} u_K = f_K$$

$$u_K = \sin 2\pi K x$$

$$f_K = -4\pi^2 K^2 \sin 2\pi K x$$

Thm Let $f(x) \in C^1[\overset{0,\,2\pi}{\cancel{-\infty,\infty}}]$. Then f has the Fourier series rep.

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{w=-\infty}^{\infty} \hat{f}(w) e^{iwx}$$

where $$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-iwx} f(x)\,dx$$

$$2i \sin(z) = \exp(iz) - \exp(-iz)$$
$$2\cos(z) = \exp(iz) + \exp(-iz)$$

Basics of Fourier

Func prod $$(f,g) = \int_0^{2\pi} f \bar{g}\,dx$$

$$\|f\| = \sqrt{(f,f)}$$

Inner product is bilinear

$$(f,g) = \overline{(g,f)}$$

$$(f+g,h) = (f,h) + (g,h)$$
$$(f,\lambda g) = \lambda (f,g)$$

Useful Inequalities

$$|(f,g)| \le \|f\| \cdot \|g\|$$
$$\|f+g\| \le \|f\| + \|g\|$$
$$\big|\,\|f\| - \|g\|\,\big| \le \|f-g\|$$

$$(f,g) \le \delta \|f\|^2 + \frac{1}{\delta}\|g\|^2, \quad \delta > 0$$

For Matrices/Operators

$$|Au| \le |A|\,|u|, \quad |A| = \max_{u \ne 0} \frac{|Au|}{|u|}$$
$$|A+B| \le |A| + |B|$$
$$|AB| \le |A|\,|B|$$

Spectral Radius $\rho(A) = \max|\lambda_i|$, $\rho(A) \le |A|$

Lemma Orthonormality

$$\left(\frac{1}{\sqrt{2\pi}} e^{inx}, \frac{1}{\sqrt{2\pi}} e^{imx}\right) = \begin{cases} 1 & n = m \\ 0 & n \ne m \end{cases}$$

Thm Parseval's Theorem

Let $A, B \in L^2([0,2\pi])$

$$A(x) = \sum_{n=-\infty}^{\infty} a_n \exp inx$$

complex conjugate

$$\overline{B}(x) = \sum_{m=-\infty}^{\infty} \overline{b_m} \exp -imx$$

Then $$\sum_{n=-\infty}^{\infty} a_n \overline{b_n} = \frac{1}{2\pi} \int_0^{2\pi} A(x) \overline{B(x)} \, dx$$

Pf $$\int_0^{2\pi} A \overline{B} dx = \sum_{n,m=-\infty}^{\infty} a_n \overline{b_m} \left(\exp(inx), \exp(imx) \right)$$

Kronecker delta

$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$$= \sum_{n,m} a_n \overline{b_m} \, 2\pi \, \delta_{nm}$$

$$= \sum_{n} a_n \overline{b_m} \, 2\pi$$

First-Order Wave Eqn

$\boxed{\text{Simplest hyperbolic problem}}$

$$(\star)\ \begin{cases} \partial_t u - \partial_x u = 0 \\ u(x, t=0) = f(x) \end{cases}$$

Expand f in a single Fourier mode

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp(iwx)\, \hat{f}(w)$$

Make ansatz

$$u(x) = \frac{1}{\sqrt{2\pi}} \exp(iwx)\, \hat{u}(w,t)$$

Substituting into $(\star)$ yields the Fourier transform of $(\star)$

$$\begin{cases} \dfrac{d\hat{u}}{dt} = iw\hat{u} \\ \hat{u}(t=0, w) = \hat{f}(w) \end{cases} \Rightarrow \hat{u}(w,t) = \exp(iwt)\,\hat{f}(w)$$

So that, after back substitution and by superposition

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_{w=\infty}^{\infty} \exp[iw(x+t)] = f(x+t)$$

$\Rightarrow$ Solutions consist of translating IC

Our first energetic principle

For all t

$$\|u(\cdot,t)\|^2 = \int u^2 dx = \sum_{w=-\infty}^{\infty} |\exp(iwt)\hat{f}(w)|^2$$

$$= \sum_{w} |\hat{f}(w)|^2$$

$$= \int f^2$$

$$= \|f\|^2$$

We call $\|u\|^2$ the energy of u

Structure we want to preserve

- energy conservation
- speed of propagation

  Along $x+t$ characteristics

  $u(x+t) = \text{const.}$

Periodic Grid functions

grid

Let $$h = \frac{2\pi}{N+1}$$

$$x_j = jh$$

grid function

$$u_j := u(x_j) = u(x_j + 2\pi) = u_{j+N+1}$$

$$(uv)_j = u_j v_j \qquad (u+v)_j = u_j + v_j$$

Translation operator

~~Def~~

The linear operator satisfying:

$$(Ev)_j = v_{j+1} \qquad (Ev)^{-1}_j = v_{j-1}$$

$$(Ev)^p_j = v_{j+p} \qquad (E^0 v)_j = v_j$$

First-order Difference operators

$$D_+ = (E - E^0)/h$$

$$D_- = (E^0 - E^{-1})/h = E^{-1} D_+$$

$$D_0 = \frac{E - E^{-1}}{2h} = \frac{1}{2}(D_+ + D_-)$$

Consider their action on a Fourier mode e^{iwx}

$$h D_+ e^{iwx} = \cancel{(e^{iwh} - 1) e^{iwx}}$$

$$= e^{iw(h+x)} - e^{iwx}$$

$$= (e^{iwh} - 1)\, e^{iwx}$$

$$= (iwh + O(w^2 h^2))\, e^{iwx}$$

Similarly

$$h D_- e^{i\omega x} = (1 - e^{-i\omega h}) e^{i\omega x}$$
$$= (i\omega h + O(\omega^2 h^2)) e^{i\omega x}$$

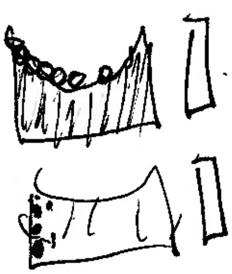
$2i \sin x = \frac{e^{ix} - e^{-ix}}{\cancel{2i}}$

$$h D_0 e^{i\omega x} = i \sin(\omega h) e^{i\omega x}$$
$$= (i\omega h + O(\omega^3 h^3)) e^{i\omega x}$$

$$|(D^+ - \partial_x) e^{i\omega x}| = O(\omega^2 h)$$
$$|(D^- - \partial_x) e^{i\omega x}| = O(\omega^2 h)$$
$$|(D^0 - \partial_x) e^{i\omega x}| = O(\omega^3 h^2)$$

Higher Order Derivatives follow from products of D_+, D_-, D_0

e.g. $(D_+ D_- v)_j = h^{-2}(v_{j+1} - 2v_j + v_{j-1})$



Claim: D operators commute

Pf
$$D_1 = \sum_i \alpha_i E_i$$
$$D_2 = \sum_j \beta_j E_j$$
$$D_1 D_2 = \sum_{ij} \alpha_i \beta_j E_{i+j} = D_2 D_1$$

Finally → Finite Differences

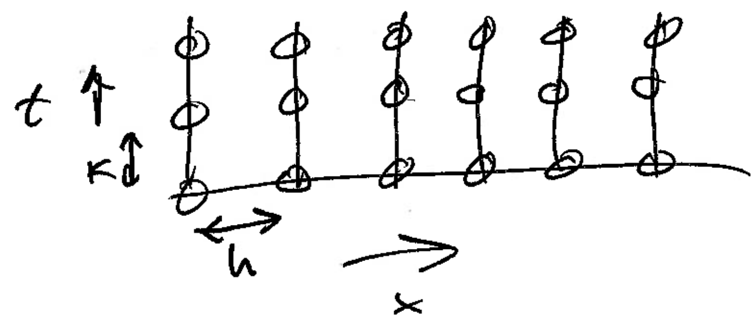
Let $h = \frac{2\pi}{N+1}$ grid size $k \ll 1$ timestep

Build a grid $(x_j, t_n) = (jh, nk)$

Discretize transport PDE Naively

$$v_j^{n+1} = (I + kD_0)v_j^n$$
$$:= Qv_j^n$$

timestep operator



To analyze, again consider a single mode but now on grid

$$f_j = \frac{1}{\sqrt{2\pi}} \exp(i w x_j)\, \hat{f}(w)$$

Ansatz

$$v_j^n = \frac{1}{\sqrt{2\pi}} \hat{v}^n(w) \exp(i w x_j)$$

Substituting into our update formula, $\lambda = \frac{k}{h}$

$$\exp(iwx_j)\,\hat{v}^{n+1}(w) = \left[\exp(iwx_j) + \frac{\lambda}{2}\left(\exp(iwx_{j+1}) - \exp(iwx_{j-1})\right)\right]\hat{v}^n(w)$$

simplifying → $\hat{v}^{n+1}(w) = (1 + i\lambda \sin(wh))\,\hat{v}^n(w)$

Define $\xi = wh$

$$\hat{v}^{n+1} = \hat{Q}\,\hat{v}^{n}$$

$$\hat{Q} = 1 + i\lambda \sin\xi$$

← "symbol" of discrete operator or "amplification factor"

Taking multiple steps

$$\hat{v}^{n}(w) = \hat{Q}^{n}\,\hat{v}^{0}(w) = \hat{Q}^{n}\,\hat{f}(w)$$

$$\Rightarrow v_j^n = \frac{1}{\sqrt{2\pi}}\left(1 + i\frac{k}{h}\sin(wh)\right)^n \exp(iwx_j)\,\hat{f}(w)$$

is the <u>exact</u> expression for the FD solution

<u>Convergence</u>

How does this behave as $h, k \downarrow 0$

$$\left(1 + i\frac{k}{h}\sin(wh)\right)^n = \left(1 + iwk + O(kh^2w^3)\right)^n$$

Reverse Taylor $$= \left(\exp(iwk) + O(k^2w^2 + kh^2w^3)\right)^n$$

Factor out $$= \left(1 + O(kw^2 + h^2w^3)\,t^n\right)e^{iwt_n}$$

$$\Rightarrow v_j^n = \frac{1}{\sqrt{2\pi}}\left(1 + O(kw^2 + h^2w^3)\,t_n\right)\exp(iw(x_j + t_n))\,\hat{f}(w)$$

Two limits

good!

$$\lim_{k,h\to 0} v_j^n = u(x_j, t_n)$$

Now fix $\lambda = \frac{k}{h} > 0$

$$\hat{Q} \sim \left(1 + ci\frac{k}{h}\right)^n$$

$$\hat{v}_j^n = (1 + ci\lambda)^{t_n/k} \cdot \hat{f}(w)$$

$$\lim_{k\downarrow 0} \hat{v}_j^n = \infty$$

Important Notes