Mon 1-27. Zoom lecture

Material

Gustafsson Ch 1, 2

Last time

- Introduction to finite differences for transport eqn
- Stability analysis and unconditional instability for explicit Euler
- True dynamics vs FD dynamics

Today

- Revisit the trigonometric interpolant
- Stable schemes for transport
- Our first learning problem

$\boxed{\text{STABILITY!}}$

Before we talked about grid functions and interpeted finite differences in terms of how they act on individual modes. Now we will formalize that.

Define the discrete inner product

$$(u,v)_h = \sum_{j=0}^{N} \bar{u}_j v_j h$$ which induces the norm

$$\|u\|_h^2 = (u,u)_h$$

All inequalities introduced last time hold analagously

$$|(u,v)_h| \leq \|u\|_h \|v\|_h$$

$$|(u,av)_h| \leq \|a\|_\infty \|u\|_h \|v\|_h \quad , \quad \|a\|_\infty = \max_j |a_j|$$

$$\|u+v\|_h \leq \|u\|_h + \|v\|_h$$

$$\left| \|u\|_h - \|v\|_h \right| \leq \|u-v\|_h$$

Stability Motivation

- FD demo
- $v_j^{n+1} = Q v_j^n$

  Amplification operator

- Distinguish true evolution from discrete FD evolution
- What happens to truncation error?
  - Lives in Q
  - Understand $\|Q\| \to$

$\|Q\| = 1$	$\Rightarrow$	Discrete conservation
$\|Q\| > 1$		Growth (bad)
$\|Q\| < 1$		Artificial dissipation (maybe bad/good)

To understand how continuous functions act when evaluated on grid points, take $u \in C^k$ and its restriction onto the grid $u(x_j)$

Then $\lim_{h \to 0} (u,v)_h = (u,v)$, $\lim_{h \downarrow 0} \|u\|_h^2 = \|u\|^2$

We are now equipped to analyze F.D. operators. We'll come at it in 2 ways

$$\|Q\|_h := \sup_{u \neq 0} \frac{\|Qu\|_h}{\|u\|_h} = \sup_{\|u\|=1} \|Qu\|_h \Rightarrow \|Qu_h\| \leq \|Q\|_h \|u_h\|$$

For a single difference operator

$$\|E^p u\|_h^2 = \sum_{j=0}^{N} |u_{j+p}|^2 h = \sum_{j'=0}^{N} |u_{j'}|^2 h = \|u\|_h^2$$

Which implies $\|E^p\| = 1$

Also

$$\|D_+ u\|_h = \frac{1}{h} \|(E - E^0)u\|_h$$
$$\leq \frac{1}{h}\|Eu\|_h + \frac{1}{h}\|Eu\|_h \quad \text{triangle inequality}$$
$$\leq \frac{2}{h}\|u\|_h$$

$$\|D_- u\|_h = \|E^{-1} D_+ u\|_h \qquad \text{Recall } D_- u = E^{-1} D_+$$
$$\leq \|E^{-1}\|_h \|D_+ u\|$$
$$\leq \frac{1}{h}\|u\|_h$$

$$\|D_0 u\|_h = \frac{1}{2h}\|E - E^{-1}\|_h \leq \frac{1}{h} \qquad \text{triangle inequality}$$

Those are actual upper bounds. But we can also bound from below. Rember the def. of the operator norm

for any Q, $\quad \|Q_h\| := \sup_{u \neq 0} \dfrac{\|Qu\|_h}{\|u\|_h}$

So for any particular $v, \neq 0$

$$\|Q_h\| \geq \frac{\|Qv\|_h}{\|v\|_h}$$

Pick $v_j = (-1)^j$

$$\Rightarrow \|v\|_h^2 = \sum_{j=0}^{N} (-1)^{2j} h$$
$$= \left(\sum_{j=0}^{N} 1\right) h$$
$$= h(N+1)$$

$$\|D_+ v\|_h^2 = h \sum_j \left[\frac{(E - E^0)}{h} v_j\right]^2$$
$$= h^{-1} \sum_j \left[(-1)^{j+1} - (-1)^j\right]^2 \qquad \text{factor out } (-1)^j$$
$$= h^{-1} \sum_{j=0}^{N} (-1)^{2j} \left[-1 - 1\right]^2$$
$$= 4h^{-1}(N+1)$$

So $\quad \|D_+\|_h^2 \geq \dfrac{\|D_+ v\|_h^2}{\|v\|_h^2} = \dfrac{4h^{-1}(N+1)}{h(N+1)} = \dfrac{4}{h^2}$

$$\|D_+\|_h \geq \frac{2}{h}$$

Now, since $\quad \dfrac{2}{h} \geq \|D_+\|_h \geq \dfrac{2}{h}$ $\qquad$ "Squeeze theorem"

$$\|D_+\|_h = \frac{2}{h}$$

Similarly $\|D_-\|_h = \frac{2}{h}$

A similar trick for D_0, picking $v_j = i^j$ $\leftarrow \sqrt{-1}$

Gives $\|D_0\|_h = \frac{1}{h}$

We now have the analysis tools to talk about the first property $\Rightarrow$ accuracy!

Now we discuss the <u>discrete</u> Fourier transform, which lets us jump from grid functions to modes. We'll skip some proofs.

<u>Thm</u> There exists a <u>unique</u> trigonometric polynomial

$$\mathrm{Int}_N(u) = \frac{1}{\sqrt{2\pi}} \sum_{w=-\frac{N}{2}}^{\frac{N}{2}} \tilde{u}(w)\, e^{iwx}$$

interpolating u, i.e

$$u_j = \mathrm{Int}_N(u_j) \quad \text{for } j=0,\dots N$$

This needs a discrete orthogonality result

<u>Lemma</u> e^{ivx} for $v = 0, \pm 1, \dots, \pm \frac{N}{2}$ are orthogonal w.r.t. the <u>discrete</u> inner product

$$(e^{ivx}, e^{i\mu x})_h = \begin{cases} 2\pi & v=\mu \\ 0 & \text{else} \end{cases}$$

<u>Pf</u> (skip) if $v=\mu$

$$(e^{ivx}, e^{ivx})_h = \sum_j h \underline{\exp(2ivx_j)}$$

$$= \sum_j h$$

$$= (N+1)h$$

$$= 2\pi$$

$e^{ikx} = \cos kx + i\sin kx$
$= 1$ if $kx_j = N\pi$ for integer N.

$x_j = \frac{N\pi}{v}$

Pick ~~N=2v~~ N

Remember $h = \frac{2\pi}{N+1}$

Thm The interpolation problem has the unique solution

$$\tilde{u}(w) = \frac{1}{\sqrt{2\pi}} (e^{iwx}, u)_h \qquad |w| \le N/2$$

Pf Assume we can interpolate, so that

$$u_j = \frac{1}{\sqrt{2\pi}} \sum_{w=-N/2}^{N/2} \tilde{u}(w) e^{iwx_j}$$

Mult both sides by $e^{-ivx} h$ and sum

$$\sum_j e^{ivx} u_j h = \frac{1}{\sqrt{2\pi}} \sum_w (e^{ivx}, e^{iwx})_h \tilde{u}(w)$$

By discrete orthogonality

$$(e^{ivx}, u)_h = \frac{1}{\sqrt{2\pi}} \sum_w \delta_{vw} 2\pi \; \tilde{u}(w)$$

$$= \sqrt{2\pi} \; \tilde{u}(v)$$

$$\Rightarrow \tilde{u}(v) = \frac{1}{\sqrt{2\pi}} (e^{ivx}, u)_h \qquad \blacksquare$$

Discrete Parseval

Thm Given $\mathrm{Int}_N u_1$, $\mathrm{Int}_N u_2$

$$(u_1, u_2)_h = \sum_{w=-\frac{N}{2}}^{N/2} \overline{\hat{u}_1(w)}\, \hat{u}_2(w)$$

We can relate the derivatives of an interpolant to the finite difference operator on gridfunctions

Thm Let $\mathrm{Int}_N u$ interp. u

Then $\|\mathrm{Int}_N u\|^2 = \|u\|_h^2$

Bridge between continuous integrals and discrete

$$\|D_+ u\|_h^2 \le \left\|\frac{d}{dx} \mathrm{Int}_N u\right\|^2$$

$$\le \frac{\pi^2}{4} \|D_+ u\|^2$$

For those interest, Gustafsson §1.3 - 1.4 has more analysis, but proofs need some trig/complex wrangling

Fundamental Theorem of Finite Differences

(Lax theorem)

Consider the abstract finite difference operator

$$v_j^{n+1} = Q v_j^n \qquad Q = \sum_\nu A_\nu(k, h) E^\nu$$

$$v_j^0 = f_j$$

If ① $\|f\|^2 < \infty$ (finite initial data)

② There exists a constant K_s independent of h, k so that

$$\sup |\hat{Q}^n| \leq K_s \qquad \text{(stability)}$$

③ For any ω

$$\lim_{k,h \to 0} \sup_{0 \le t_n < T} |\hat{Q}^n(\xi) - e^{i\omega t_n}| = 0$$

(consistency)

Then the FD scheme converges

$$\lim_{k,h \to 0} \sup_{0 \le t_n \le T} \|u(\cdot, t_n) - \mathrm{Int}_N(v_j^n)\| = 0$$

skip proof, discuss implications for code development