Lecture 3 2/3/25

Recall the Lax Equivalence Theorem

1. $\|S\|^2 < \infty$
2. $\sup |\hat{Q}^n| \le K_S$
3. $\lim\limits_{K,h\to 0} \sup\limits_{0\le t_n \le T} |\hat{Q}^n(\xi) - e^{i w t_n}| = 0$

~~For~~ $\Rightarrow \lim\limits_{K,h\to 0} \sup\limits_{0\le t_n\le T} \|u(\cdot,t_n) - \mathrm{Int}_N(v_j^n)\| = 0$

For finite IC, stability + consistency imply convergence

Today we'll consider the learning problem

$$u^{target}(x,t;k) = \sin(k(x-t))$$

$$\partial_t u = \mathcal{L}(u;\theta)$$

$$\min_\theta \|\partial_t u^{target} - \mathcal{L}(u;\theta)\|^2$$

And how to approach 2 + 3

Recall that the explicit Euler + centered diff. scheme for transport eqn is <u>unconditionally unstable</u>

i.e. $\partial_t u = \partial_x u$

$$\frac{v_j^{n+1} - v_j^n}{k} = D_0 v_j^n$$

For a single mode w

$$\Rightarrow v_j^n = \frac{1}{\sqrt{2\pi}} \left(1 + i \frac{k}{h} \sin(wh)\right)^n \times \exp(i w x_j) \hat{f}(w)$$

want indep. of k, h

$$|\hat{Q}| \le K_s \iff \left|1 + i \frac{k}{h} \sin(wh)\right| \le K_s$$

$$\left|1 + i \frac{k}{h} \sin(wh)\right| \le 1 + \frac{k}{h} \left| i \sin(wh)\right| \le 1 + \frac{k}{h}$$

- For any finite k/h, it will grow.
- Consistent, but not stable
- Can add <u>artificial viscosity</u> to control growth

$$\partial_t u = \partial_x u + \sigma h \partial_{xx} u$$

as $h \downarrow 0$ recover true eqn, so OK for consistency

$$-\frac{v_j^{n+1} - v_j^n}{K} = D_0 v_j^n + \sigma h D_+ D_- v_j^n$$

- Choose σ, K, h so that $|\hat{Q}| \leq 1$

- Recall $h D_0 e^{iwx} = i \sin(wh) e^{iwx}$

$$h^2 D_+ D_- e^{iwx} = -4 \sin^2\left(\frac{wh}{2}\right) e^{iwx}$$

So we could show

$$\hat{Q} = 1 + i\lambda \sin\xi - 4\sigma\lambda \sin^2\frac{\xi}{2}$$

$\lambda = K/h$

$\xi = wh$

$|a+ib|^2 = a^2+b^2$

$$|\hat{Q}|^2 = \left(1 - 4\sigma\lambda \sin^2\frac{\xi}{2}\right)^2 + \lambda^2 \sin^2\xi$$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

Expand powers

$$|\hat{Q}|^2 = 1 - 8\sigma\lambda\sin^2\frac{\xi}{2} + 16\sigma^2\lambda^2\sin^4\frac{\xi}{2} + 4\lambda^2\sin^2\frac{\xi}{2}\cos^2\frac{\xi}{2}$$

$$= \text{"} - \text{"} + \text{"} + 4\lambda^2\sin^2\frac{\xi}{2}\left(1 - \sin^2\frac{\xi}{2}\right)$$

Collect powers of $\sin\frac{\xi}{2}$

$$= 1 - \left(8\sigma\lambda - 4\lambda^2\right)\sin^2\frac{\xi}{2} + \left(16\sigma^2 - 4\right)\lambda^2\sin^4\frac{\xi}{2}$$

We can make this ≤ 1 in two ways

(1) Make both terms negative

$$8\sigma\lambda - 4\lambda^2 \geq 0 \quad , \quad 16\sigma^2 - 4 \leq 0$$

$$\sigma \geq \frac{\lambda}{2} \qquad \sigma \leq \frac{1}{2}$$

So pick $0 < \lambda \leq 2\sigma \leq 1$

(2) ~~For small ξ, $\sin\xi \sim \xi$~~

Take $\sin\frac{\xi}{2} = 1$

$$|\hat{Q}|^2 = 1 - (8\sigma\lambda - 4\lambda^2) + (16\sigma^2 - 4)\lambda^2 \leq 1$$

$$0 \leq 8\sigma\lambda - 4\lambda^2 - 16\sigma^2\lambda^2 + 4\lambda^2$$

So we can have $\sigma \geq \frac{1}{2}$ if

$$2\sigma\lambda \leq 1$$

This analysis gives two schemes:

(1) Lax-Friedrichs Method $\left(\sigma = \frac{h}{2k} = \frac{1}{2\lambda}\right)$

$$v_j^{n+1} = (I + kD_0)v_j^n + \frac{1}{2}h^2 D_+ D_- v_j^n$$

(2) Lax-Wendroff Method $\left(\sigma = \frac{k}{2h} = \frac{\lambda}{2}\right)$

$$v_j^{n+1} = (I + kD_0)v_j^n + \frac{1}{2}k^2 D_+ D_- v_j^n$$

- Those give stability by adding non-physical terms, this is our first example of <u>numerical stabilization</u>

- Often we can just add a dissipative term w/ a fudge factor in front

- Instead of adding terms we can use <u>implicit</u> schemes

$$\frac{v_j^{n+1} - v_j^n}{k} = D_0 v_j^{n+1}$$

$$(I - kD_0) v_j^{n+1} = v_j^n$$

$$\hat{Q} = \left[1 - i\lambda \sin\xi\right]^{-1} \leq 1$$

for <u>any</u> λ, σ!

This is called unconditional stability, but needs us to solve a linear system to update

One special implicit scheme is a trapezoid rule in time

<u>Crank-Nicholson method</u> $\frac{v_j^{n+1} - v_j^n}{k} = \frac{1}{2} D_0 v_j^{n+1} + \frac{1}{2} D_0 v_j^n$

which has $|Q| = 1$ exactly

Ref "On generalized moving least squares and diffuse derivatives" Mirzaei et al.

Now that we have a handle on stability tools we can turn to how to build accuracy guarantees

Assume we want to approximate a differential operator D^α. In <u>multi-index notation</u> α is a tuple denoting mixed derivatives $(e.g. \alpha = (1,2) \Rightarrow D^\alpha = \partial_x \partial_{yy})$ and $|\alpha|$ the order of the derivative.

$D_h^\alpha v_i = \sum_j s_{ji} v_j$ is a generic stencil

Thm If

polynomial reproduction

① $\sum_j s_{ji} p_j = D^\alpha p(x_i)$ for a m^{th} order polynomial p

② $\sum |s_{ji}| \leq C h^{-|\alpha|}$

Then $\| D_h^\alpha u_i - D^\alpha u_i \|_{L^\infty} \leq C h^{m+1-|\alpha|}$

$$D_h^\alpha u_i = \sum_j S_{ji}^\alpha u_j \qquad D_h \approx D^\alpha$$

① $\sum_j S_{ji} P_j = D_h P(x_i)$

② $\sum_j |S_{ji}| < C_1 h^{-|\alpha|}$

Pf/ Let $p \in P_m$ be m^{th} order polynomial

$$|D_h u - Du| \le |Du - Dp| + |Dp - D_h u|$$

$$= \quad '' \quad + \sum_j S_{ji} (p_j - u_j)$$

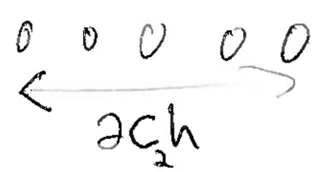
$$\le \quad '' \quad + \sum_j |S_{ji}|\, |p_j - u_j|$$

$L^\infty(u) = \max_{x \in \Omega} u(x)$

$$\le \|Du - Dp\|_{L^\infty} + \|u - p\|_{L^\infty} \sum |S_{ji}|$$

$$\le \|Du - Dp\|_{L^\infty} + C_1 h^{-|\alpha|} \|u - p\|_{L^\infty}$$

Choosing p as the taylor expansion of u at x_i



$$\|u - p\|_{L^\infty} \le C_2 h^{m+1} |u|_{C^{m+1}}$$

$$\|D^\alpha u - D^\alpha p\|_{L^\infty} \le C_3 h^{m+1-|\alpha|} |u|_{C^{m+1}}$$