Lecture 7
2/19

To spatially discretize Lagrangian

$$S = \iint \frac{1}{2}(\partial_t u)^2 - \frac{c^2}{2}(\partial_x u)^2 \, dx\, dt$$

Define piecewise constant extension of a grid function

$$\phi_i(x) = \mathbb{1}_{(x-\frac{h}{2},\, x+\frac{h}{2})}(x) = \begin{cases} 1 & x \in (x-h/2,\, x+h/2) \\ 0 & \text{else} \end{cases}$$

$$u(x,t) = \sum_{i=1}^{N} u_i(t)\, \phi_i(x)$$

$$\partial_x u(x,t) = \sum_{i=1}^{N} \underbrace{\frac{1}{h}\left(\alpha u_{i-1}(t) + \beta u_i(t) + \gamma u_{i+1}(t)\right)}_{D_h u_i} \phi_i(x)$$

Note that $\int u \, dx = \sum_i h u_i(t)$

and selecting $q_i = u_i$, we obtain the Lagrangian

$$L = \sum_i \frac{1}{2} \dot{q}_i^2 h - \frac{c^2}{2} h \, D_h q_i^2$$

In light of Noether's theorem

① Trivial time invariance (time only enters in $\dot{q}_i$)

$$\Rightarrow H = \partial_{\dot{q}} L \cdot \dot{q} - L \quad \text{is conserved}$$

Where the conjugate momentum

$$p_i = \partial_{\dot{q}_i} L = \dot{q}_i h$$

② Translation invariance $q \to q + \delta q$

only if $D_h \delta q = 0$ for a constant shift vector δq

$$\text{Const} = \sum_i p_i = \sum_i h \dot{q}_i$$

To take variations in this spatially discretized setting we need a <u>discrete</u> version of integration by parts

<u>Lemma</u> Consider two <u>periodic</u> grid functions u_i, v_i satisfying $u_0 = u_N$, $v_0 = v_N$

$$\sum_{i=1}^{N} x_i y_{i+\alpha} = \sum_{j=1+\alpha}^{(N+\alpha)\%N} x_{j-\alpha} y_j \qquad (j = i+\alpha)$$

$$= \sum_{j=1}^{N} x_{j-\alpha} y_j \qquad \text{(shift limits / addition is commutative)}$$

$$= \sum_{i=1}^{N} x_{i-\alpha} y_i$$

Compactly in terms of shift operators

$$\langle x, E^{\alpha} y \rangle = \langle E^{-\alpha} x, y \rangle$$

Consider now a given stencil operator $D_h u_i = \sum_{k=-m}^{m} c_k E^k u_i$

<u>Def</u> Define the adjoint operator $D_h^* u_i = \sum_{k=-m}^{m} c_{-k} E^k u_i$

Then $\langle D_h^* u, \delta u \rangle = \langle u, D_h \delta u \rangle$

Pf

$$D_h u = \sum_{k=-m}^{m} c_k E^k u_i$$

$$\langle x, D_h u \rangle = \sum_{k=-m}^{m} \langle x, c_k E^k u_i \rangle$$

$$= \sum_{k=-m}^{m} c_k \langle x, E^k u \rangle$$

$$= \sum_{k=-m}^{m} c_k \langle E^{-k} x, u \rangle$$

$$= \cancel{\sum u} \left\langle \sum_k c_k E^{-k} x, u \right\rangle$$

$$= \left\langle \sum_k c_{-k} E^k x, u \right\rangle$$

$$= \langle D_h^* x, u \rangle \qquad \blacksquare$$

Before we attempt to tackle the full Lagrangian, lets see how the variations look / mirror the cart. setting

$$F[u] = \frac{1}{2}\int \nabla u^2$$

$$\begin{aligned}
(\delta_u F, \delta u) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(F[u+\varepsilon\delta u] - F[u]\right) \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\int\left[\frac{1}{2}\nabla u^2 + \varepsilon \nabla u\cdot\nabla\delta u + \varepsilon^2\frac{1}{2}\nabla\delta u^2 - \frac{1}{2}\nabla u^2\right]dx \\
&= \int \nabla u \cdot \nabla \delta u \\
&= (\nabla u, \nabla \delta u) \\
&= -(\nabla\cdot\nabla u, \delta u)
\end{aligned}$$

$$\Rightarrow \delta_u F = -\nabla^2 u$$

Discrete

$$F_h[u] = \frac{1}{2}\int\Big(\sum_i D_0 u_i\, \phi_i(x)\Big)^2 dx, \quad D_0 u_i = \frac{u_{i+1} - u_{i-1}}{2h}$$

$$= \sum_i \frac{h}{2}(D_0 u_i)^2$$

$$\begin{aligned}
\langle \delta_{u_i} F_h[u], \delta u_i\rangle &= \lim_{\varepsilon \downarrow 0}\frac{1}{\varepsilon}\sum_i\left[\frac{h}{2}(D_0 u_i + \varepsilon D_0 \delta u_i)^2 - \frac{h}{2} D_0 u_i^2\right] \\
&= \sum_i D_0 u_i \cdot D_0 \delta u_i \\
&= \langle D_0 u, D_0 \delta u\rangle = \langle \underline{D_0^* D_0} u, \delta u\rangle
\end{aligned}$$

Now we return to the discrete wave eqn Lagrangian

Let $D_h u = \frac{1}{h} \sum_{k=-m}^{m} c_k E^k u_i$

ex $m=1$, $c_{-1}=-1$, $c_0=0$, $c_1=1 \Rightarrow D_h = D_0$

Recall

$$S(q,\dot{q}) = \int \sum_i \Big[\underbrace{\tfrac{1}{2} \dot{q}_i^2 h}_{S_1} - \underbrace{\tfrac{c^2}{2} (D_h q)_i^2 h}_{S_2} \Big] dt$$

$$(\delta_{q_i} S_1, \delta q_i) = \int \sum_i \dot{q}_i \cdot \delta \dot{q}_i h \, dt$$

$$= \int -\sum_i \ddot{q}_i h \;\; \delta q_i \, dt$$

$$\Rightarrow \delta_{q_i} S_1 = -\ddot{q}_i h$$

$$(\delta_{q_i} S_2, \delta q_i) = -\int \sum_i c^2 D_h q_i \cdot D_h \delta q_i h \, dt$$

$$= -\int c^2 \langle D_h q, D_h \delta q \rangle h \, dt$$

$$= -\int c^2 \langle D_h^* D_h q, \delta q \rangle h \, dt$$

$$\Rightarrow \delta_{q_i} S_2 = -c^2 h D_h^* D_h q$$

$$\delta_{q_i} S = \delta_{q_i} S_1 + \delta_q S_2 = 0$$

$$\boxed{\ddot{q}_i = -c^2 D_h^* D_h q_i}$$

Remark • While tedious, no thought was needed to reach this point - just careful application of calculus

• We now have two constraints for a good stencil

① $q \to q + \delta q$ invariance $\Rightarrow \sum' c_k = 0$

i.e $D_h \delta q = \sum_k' c_k E^k \delta q$

$= \delta q \left(\sum_k' c_k\right)$ if $\delta q = \text{const}$

$= 0 \Rightarrow \sum' c_k = 0$

② Does $-D_h^* D_h q$ give a stable prediction of ∇_h^2?

- Recall from a few lectures back, this is true if

$D_h^* D_h p = \nabla^2 p$ for any quadratic p

• Let's translate that into a system of algebraic constraints.

$$D_h u_i = \frac{1}{h} \sum_j c_j u_{i+j}$$

$$D_h^* v_j = \frac{1}{h} \sum_K c_{-K} v_{j+K}$$

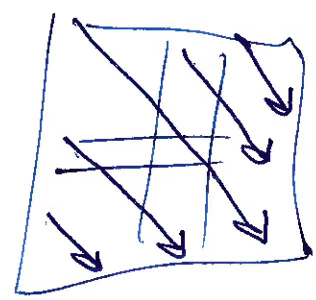
$$D_h^* \circ D_h u_i = \frac{1}{h^2} \sum_{j,K} c_j c_{-K} u_{i+j+K}$$

To do arithmetic, set

$$c_{-1} = \alpha \qquad c_0 = \beta \qquad c_1 = \gamma$$

$$\begin{array}{cc|c|c|c|}
 & & & j= & \\
 & & -1 & 0 & 1 \\
\hline
 & 1 & \alpha^2 & \alpha\beta & \alpha\gamma \\
\hline
k= & 0 & \beta\alpha & \beta^2 & \beta\gamma \\
\hline
 & -1 & \gamma\alpha & \gamma\beta & \gamma^2 \\
\hline
\end{array}$$

Read of diagonals



$$\Rightarrow D_h^* \circ D_h u_i = -\frac{c^2}{h^2}\Big[(\alpha\gamma)\, u_{i+2} + (\alpha\beta + \gamma\beta)\, u_{i+1} + (\alpha^2 + \beta^2 + \gamma^2)\, u_i + (\beta\alpha + \beta\gamma)\, u_{i-1} + (\alpha\gamma) u_{i-2} \Big]$$

① Noether constraint

$$\alpha + \beta + \gamma = 0$$

Note to self
Leave stencil up on board

② Constant Reproduction

$$2\alpha\gamma + 2\beta(\alpha+\gamma) + \alpha^2 + \beta^2 + \gamma^2 = 0$$

③ Linear Reproduction (Take $u_i = 0$, $u_{i+1} = h$, etc)

$$-2\alpha\gamma h - \beta(\alpha+\gamma)h + 0 + \beta(\alpha+\gamma)h + 2\alpha\gamma h = 0$$

(Automatically satisfied)

④ Quadratic Reproduction

$$\alpha\gamma(2h)^2 + \beta(\alpha+\gamma)h^2 + 0 + \beta(\alpha+\gamma)h^2 + \alpha\gamma(2h)^2 = -2$$

$$8\alpha\gamma + 2\beta(\alpha+\gamma) = -2$$

Solving (1-4) is non-unique, but try some simplifications

Ⓐ Assymetric stencil

$\gamma = 0$

① $\Rightarrow \alpha = -\beta$

④ $\Rightarrow -2\beta^2 = -2$

$\beta = 1, \alpha = -1$

$$\Rightarrow D_h = -\frac{1}{h^2}\left(-g_{i+1} + 2g_i - g_{i-1}\right)$$

$$= D_+ D_- = D_- D_+$$

$\boxed{\text{We recover 3-pt laplacian!}}$

Ⓑ Symmetric

$$\alpha = \gamma = -2\beta$$

④ $\Rightarrow 8\alpha^2 - 2\alpha^2 = -2$

$$\alpha = \sqrt{-\frac{1}{3}}$$

$\boxed{\text{No real valued solns!}}$

(c)

$\alpha = -\gamma$

(1) $\Rightarrow \beta = 0$

(4) $-8\alpha^2 = -2$

$\alpha = \frac{1}{2}$

(2) $-2\alpha^2 + 2\alpha^2 + \beta^2 = 0$

$$D_h u = \frac{1}{h^2}\left[-\frac{1}{4} g_{i-2} + \frac{1}{2} g_i - \frac{1}{4} g_{i+2}\right]$$