Exercise 1

Prove that $\forall\, v \in \mathbb{R}^n$

$$c\,\|v\|_\infty \leq \|v\|_2 \leq C\,\|v\|_\infty$$

- This is called <u>norm equivalence</u> and means that up to a scale factor the max-norm is the same as ℓ^2-norm

<u>PF</u>

$$\|v\|_2^2 = \sum_i v_i^2$$

$$= v_{max}^2 + \sum_{i \neq max} v_i^2$$

$$\geq v_{max}^2$$

$$= \|v\|_\infty^2$$

other direction

$$\|v\|_2^2 = \sum_i v_i^2$$

$$\leq \sum_i v_{max}^2$$

$$= v_{max}^2 \left(\sum_{i=1}^{N} \right)$$

$$= N \, \|v\|_\infty^2$$

$$\Rightarrow \|v\|_2 \leq \sqrt{N} \, \|v\|_\infty \quad \blacksquare$$

Exercise 2

Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$

Compute $\frac{\partial}{\partial x} \|Ax\|^2$ w/ Einstein notation

Pf $\|Ax\|^2 = A_{ij} x_j A_{ik} x_k$

Differentiate w/ product rule

$$\frac{\partial}{\partial x_a}(x_j x_k) = x_j \delta_{ak} + x_k \delta_{aj}$$

$$\Rightarrow \frac{\partial}{\partial x_a} \|Ax\|^2 = A_{ij} A_{ik} (x_j \delta_{ak} + x_k \delta_{aj})$$

$$= A_{ij} A_{ia} x_j + A_{ia} A_{ik} x_k$$

$$= 2 A_{ij} A_{ia} x_j$$

$$= 2 A^T A x$$

Combine dummy index

$\blacksquare$

Exercise 3

Given $a_1, \dots a_n \in \mathbb{R}$ prove

$$\left| \frac{1}{n} \sum_i a_i \right| \leq \sqrt{\frac{1}{n} \sum a_i^2}$$

i.e $|\text{mean}| < \text{RMS}$

Recall cauchy - schwartz

$$|\langle x, y\rangle| \le \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}$$

Multiply by one trick

$$\sum_i a_i = \langle a_i, \vec{1}\rangle$$

vector of all ones

$$\left|\sum a_i\right| \le \|a\|_2 \, \|\vec{1}\|_2$$

$$= \|a\|_2 \sqrt{n}$$

Divide by n

$$\left|\frac{1}{n}\sum a_i\right| \le \sqrt{\frac{1}{n}\sum a_i^2}$$

$\blacksquare$

Exercise 4

How to use Young's ineq.

"Rob Peter to pay Paul"

$$|fg| \le \frac{|f|^2}{2\epsilon} + \frac{\epsilon |g|^2}{2}$$

$2\delta \qquad\qquad 2$

- A tool to split products into sums (Young's eq. if $\delta = 1$)
- Can borrow from f to give to g

ex Assume $|g|^2 \le \varepsilon$, $|f|^2 < C$

Prove $\lim_{\varepsilon \to 0} |fg| = 0$

Pf

$$|fg| \le \frac{|g|^2}{2\delta} + \frac{\delta}{2}|f|^2$$

$$\le \frac{\varepsilon}{2\delta} + \frac{\delta C}{2}$$

We have two terms and one

[illegible]

doesn't scale w/ ε, but we can choose δ.

1) Eyeball it. $\delta = \sqrt{\varepsilon / C}$

$$|fg| < \sqrt{\varepsilon C}$$

makes them scale the same

2) A more sophisticated trick. Choose δ to make bound as tight as possible

$$\frac{d}{d\delta}\left(\frac{\varepsilon}{2\delta} + \frac{\delta C}{2}\right) = -\frac{\varepsilon}{2\delta^2} + \frac{C}{2} = 0$$

$$\frac{\delta^2 C}{\cancel{2}} = \frac{\varepsilon}{\cancel{2}} \qquad \delta^2 = \frac{\varepsilon}{C}$$

$$\boxed{\delta = \sqrt{\varepsilon / C}}$$

This is a trick Terence Tao calls an "ε of room". Build a free parameter in to an estimate and then solve for how to make it do what you want.

Exercise 5

Define the linear operator

$$A : \mathbb{R}^n \to \mathbb{R}^n$$

by $$(Ax)_i = \frac{1}{2}(x_{i-1} + x_i + x_{i+1})$$

w/ periodic BCs

Compute the matrix norm in ℓ^2, ℓ^∞

Pf Sketch matrix

$$A = \begin{pmatrix} 1 & 1 & & & 1 \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & & \ddots & \ddots & \ddots \\ 1 & & & 1 & 1 \end{pmatrix}$$

This is called circulant & pops up for periodic problems

ℓ^∞-norm $\quad \|A\|_\infty = \max_i \sum_j |A_{ij}|$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

ℓ^2-norm $\quad \|A\|_2^2 \le \|A\|_1 \|A\|_\infty$

Note $\|A\|_\infty$ is max row sum

$\|A\|_1$ is max <u>column</u> sum

$$\Rightarrow \|A\|_1 = \|A\|_\infty = 1$$

$$\|A\|_2^2 \le 1 \qquad \blacksquare$$

Why do we care?

We can estimate now

$\|Ax\|_\infty$ & $\|Ax\|_2$ <u>for any</u> <u>x</u>

Recall

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$\Rightarrow \|Ax\| \le \|A\| \, \|x\|$$

So in our example $\|A\|=1$

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$$\|Ax\| \leq \|x\|$$

and we can say that the filtered x is never bigger than the initial x (in norm).