# Lecture Notes 2/4 – Hands-on Coding

In the coming weeks we'll consider two problem flavors

IVP $$\frac{u_j^{n+1} - u_j^n}{K} = D_+D_- u\big|_j + N[u]\big|_j$$

BVP $$D_+D_- u\big|_j + N[u]_j = 0$$

- Depending on RHS for IVP being implicit/explicit will req. non linear solve
- So far we've only studied <u>linear</u> problems

## Today's Exercise

1. Learn to reliably solve non-linear systems of eqns
2. Learn to solve for a <u>linear</u> stencil

In ~ 1 week we'll put it all together

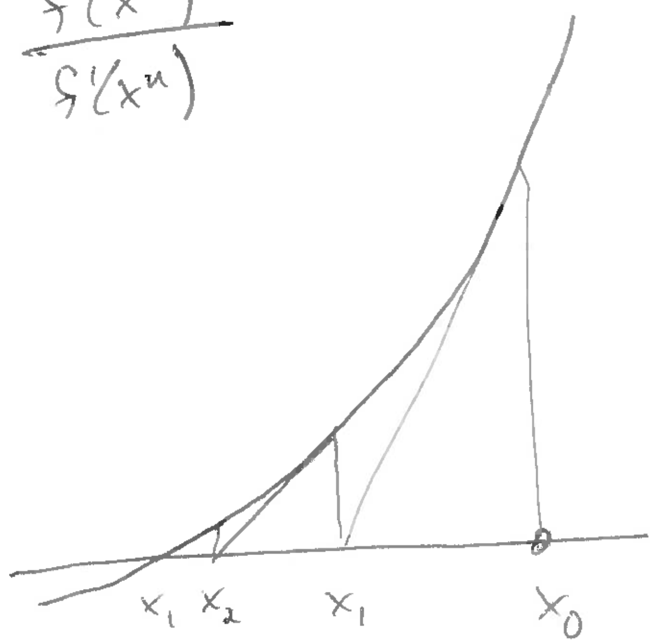
# Newtons Method -

- Find $f(x^*) = 0$
- Taylor expand around guess $x^n$

$$f(x^n) + f'(x^n)(x - x^n) = 0$$

- Solve for $x \to x^{n+1}$

$$x^{n+1} = x^n - \frac{f(x^n)}{f'(x^n)}$$




(quadratic conv.)

**Thm** Let $x^*$ solve $f$.

There exists $\delta > 0$ such that for $|x_0 - x_*| < \delta$

$$|x_{k+1} - x^*| \leq C|x_k - x_*|^2$$

**Pf** Let $e_k = x_k - x_*$

For $\xi_k \in [x_k, x_*]$

expand at $x^*$ →

$$f(x_k) = \underbrace{f(x^*)}_{=0} + f'(x^*)e_k + \frac{1}{2} f''(\xi_k)\, e_k^2$$

similarly $f'(x_k) = f'(x^*) + f''(\eta_k)\, e_k$ for $\eta_k \in [x_k, x_*]$

Write newton step minus $x^*$

$$\underbrace{x_{k+1} - x_*}_{e_{k+1}} = \underbrace{x_k - x^*}_{e_k} - \frac{f(x_k)}{f'(x_k)}$$

Substitute in Taylor series + simplify (trust me ☺)

$$e_{k+1} = e_k \left( 1 - \frac{1 + \frac{1}{2} \frac{f''(\xi_k)}{f'(x^*)} e_k^2}{1 + \frac{f''(\eta_k)}{f'(x^*)} e_k} \right)$$

For small $|e_k|$ $\quad (1+x)^{-1} \approx 1 - x + O(x^2)$

$$e_{k+1} = e_k \left( \frac{1}{2} \frac{f''(\xi_k)}{f'(x^*)} e_k + O(e_k^2) \right)$$

$|e_0| < \delta$ justifies

$$e_{k+1} = \underbrace{\frac{1}{2} \frac{f''(\xi_k)}{f'(x^*)}}_{\leq C} e_k^2 + HOT$$

## Multivariate Case

$$\Delta_n = -J^{-1}(x^n)\, F(x^n)$$
$$x^{n+1} = x^n + \Delta_n$$

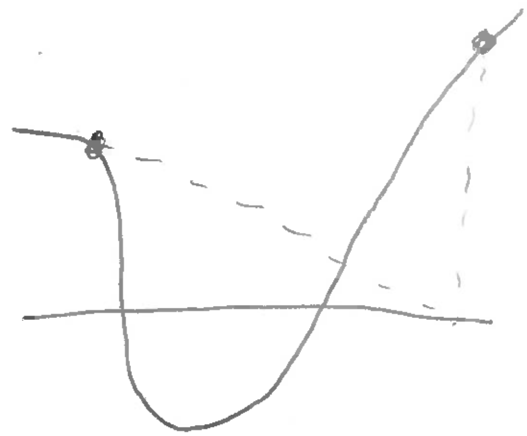
## Backtracking

Newton requires $e_0 < \delta$ (good init guess)

Error can get worse!

$$x^{n+1} = x^n + \alpha^n \Delta^n$$

$\uparrow$ relaxation



$\alpha \in [0,1]$

$\alpha \searrow 0 \rightarrow$ gradient descent

## How to pick $\alpha$? Armijo's Rule

Define $\phi(x)$ s.t. $\phi(x) = 0 \Rightarrow F(x) = 0$

(e.g. $\phi(x) = \frac{1}{2}\|F(x)\|^2$)

**Armijo** Pick maximum $\alpha_k \in [0,1]$ s.t.

$$\phi(x_k + \alpha_k \Delta_k) \leq \phi(x_k) + c_1 \alpha_k \nabla\phi(x_k)^T \Delta_k$$

$\uparrow$ typically $10^{-4}$

<u>Thm</u> For cont. $F$, + differentiable $\phi(x_{k+1}) < \phi(x_k)$ if $\Delta \neq 0$

<u>Pf</u> Consider WLOG $\phi = \frac{1}{2}\|F\|^2$

$$\nabla\phi = J(x_k)^T F(x_k)$$

$$\nabla\phi(x_k)^T \Delta_k = -\left(J(x_k)^T F(x_k)\right)^T \left(J(x_k)^{-1} F(x_k)\right)$$

$$= -\|F(x_k)\|^2$$

<u>If $J$ invertible</u>

Taylor expansion

$$\phi(x_k + \alpha_k \Delta_k) = \phi(x_k) + \underbrace{\alpha_k \nabla\phi(x_k)^T \Delta_k}_{\text{showed this is negative}} + o(\cancel{\alpha_k^2})$$

$$\leq \phi(x_k) + c_1 \alpha_k \nabla\phi(x_k)^T \Delta_k$$

for any $c_1 < 1$ ∎

Punchline

For small enough $\alpha$, relaxation guaranteed to improve error

$\alpha_k^0 = 1$, $\alpha_k^{i+1} = \varepsilon \alpha_k^i$ until $\phi$ decreases