Last time we discussed locking and showed that Stokes flow is a special limit of incompressible elasticity.

$$\nabla^2 u - \nabla p = f \qquad \text{Lagrange mult.}$$
$$\nabla \cdot u = 0 \qquad \text{incompressibility constraint}$$

This is an example of an abstract saddle point problem

$$(\star) \quad \begin{aligned} a(u,v) + b(p,v) &= L_1(v) \\ b(q,u) &= L_2(q) \end{aligned} \qquad \forall\, v \in V,\ q \in M$$

Note that this corresponds to a matrix system

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Assume a satisfies Lax-Milgram

<u>Thm</u> (A) has a unique soln if V, M satisfy an inf-sup (aka LBB) condition

<u>def</u> <u>inf-sup</u>

V & M satisfy inf-sup condition, LBB condition, or are inf-sup compatible if $\forall\ p \in M$, $\exists\ \tilde{v} \in V$ nonzero s.t

$$\frac{b(p, \tilde{v})}{\|\tilde{v}\|_V} \geq \beta \|p\|_M$$

<u>Pf</u> - Recall $Ax = b$ has a solution if $b = 0 \Rightarrow x = 0$

- Take $l_1 = l_2 = 0$, $q = p$, $v = u$
- $$\begin{aligned} a(u,u) + b(p,u) &= 0 \\ b(p,u) &= 0 \end{aligned} \rightarrow a(u,u) = 0$$
- Coercivity $0 = a(u,u) \geq \alpha \|u\|_V$
$$\Rightarrow u = 0$$
- By inf-sup, $\exists\ \tilde{v} \in V$. Take $v = \tilde{v}$
$$b(p, \tilde{v}) = 0$$
$$0 = \frac{b(p, \tilde{v})}{\|\tilde{v}\|_V} \geq \beta \|p\|_M \Rightarrow p = 0$$

∎

That's a little abstract. Lets do a concrete 1D example, for $\Omega = [0,1]$

$$-u'' = f \rightarrow \begin{cases} F = u' \\ -F' = f \end{cases}$$

Pick $F \in V$, $u \in M$, $\forall v \in V, q \in M$

$$(F,v) - (u',v) = 0$$
$$-(F',q) = (f,q)$$

If we IBP top right

$$(F,v) + (u,v') = uv|_0^1$$
$$-(F',q) = (f,q)$$

we have a saddle point problem

- $a(F,v) = (F,v)$ ← satisfies Lax conditions
- $b(u,v) = (u,v')$

Lets build an inf-sup compatible space for this problem

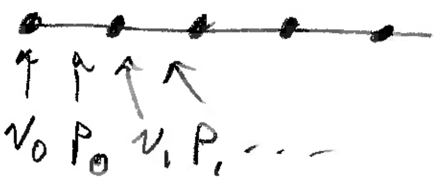
Let $M_h = \{ f \in L^2(\Omega),\ f|_e = \text{const} \}$

V_h = piecewise linears

Task: Given ~~any~~ p, build a $\tilde{v} \in V$ such that $\tilde{v}' = p$

Hint: Think FTC

Notation



Set
$$\tilde{v}_0 = 0$$
$$\tilde{v}_j = \sum_{k=0}^{j-1} p_k h$$

Then
$$\int_{e_j} \tilde{v}' \, dx = \tilde{v}_{j+1} - \tilde{v}_j = p_j h$$

This construction is building an antiderivative → easy in 1D

Note that
$$b(p, \tilde{v}) = \int p v' \, dx$$
$$= \sum_j \hat{p}_j (\tilde{v}_{j+1} - \tilde{v}_j)$$
$$= \sum_j \hat{p}_j^2 h$$
$$= \| p \|_{L^2}^2$$

And $|v|_{H_1}^2 = \sum_{e_j} \int_{x_j}^{x_{j+1}} (v')^2 \, dx$

$$= \sum_{e_j} \int_{x_j}^{x_{j+1}} p^2 \, dx$$

$$= \|p\|_{L^2}^2$$

So $\dfrac{b(p,\tilde{v})}{|v|_{H_1}} = \dfrac{\|p\|_{L^2}^2}{\|p\|_{L^2}} = \|p\|_{L^2}$

$\Rightarrow$ inf-sup w/ $\beta = 1$

norms $V = H_1$ seminorm
$M = L^2$

Thoughts on inf-sup

- it comes from FTC and building anti derivative $\rightarrow$ can we do this in higher dim?
- If we pick a bigger $V_{big} \supseteq V$, V_{big} is also inf-sup compatible
  - $\hookrightarrow$ is V big enough to find an anti derivative?
- Formally, V, M are surjective under derivative

Break to revisit vector calc

Objective (until next week) steady state dynamics

$$Au + \varepsilon N[u] = f$$

We'd like to pose this in a conservative form

$$\nabla \cdot F = f$$ ← exact! conservation

$$F = -\nabla u + N[u]$$

Then <u>any</u> learned model will be conservative

if $f = 0$

$$0 = \int \nabla \cdot F \, dx = \oint F \cdot dA$$

"goes into = goes outta"

Other flavors of conservation

$$\int \nabla \phi \cdot dl = \phi_j - \phi_i$$

$$\int \nabla \times \psi \cdot dA = \int \psi \cdot dl$$

$$\int \nabla \cdot F \, dV = \int F \cdot dA$$

$$\boxed{\int_{\Omega} dF = \int_{\partial \Omega} F}$$

Generalized Stokes Theorem

Let $d_0 = \nabla$, $d_1 = \nabla\times$, $d_2 = \nabla\cdot$

Recall $\nabla\times\nabla = \nabla\cdot\nabla\times = 0 \rightarrow \boxed{d_{k+1} d_k = 0}$

Lets build a set of FEM spaces to treat any of these

$$W_0 \xrightarrow{d_0} W_1 \xrightarrow{d_1} W_2 \xrightarrow{d_2} W_3$$

→ Define first

Theorem The saddle-pt problem

$$(F, v) - (du, v) = 0$$

$$\star\star \quad (q, d^* F) = (f, q)$$

Def d^* is the adjoint AKA what comes out of IBP

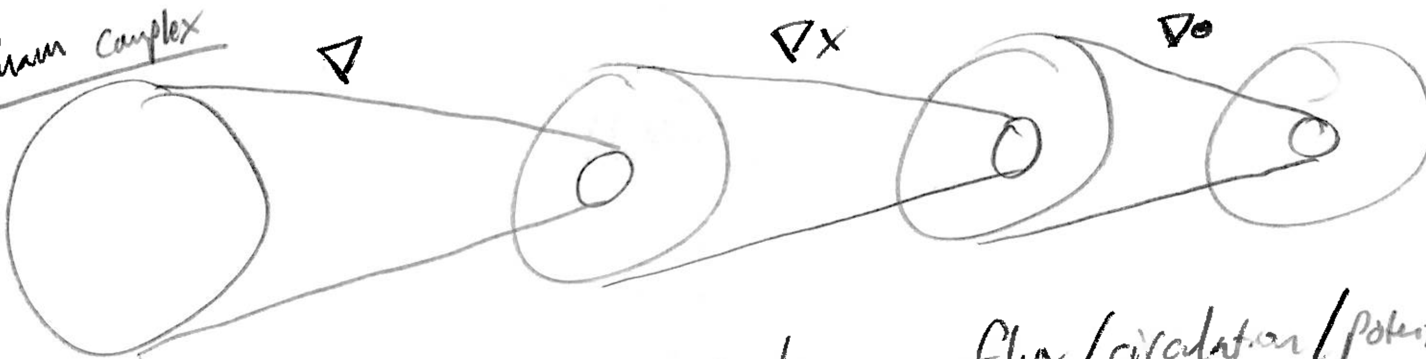
ex $(\nabla u, v) = -(u, \nabla\cdot v)$

$$\underbrace{\nabla u}_{d_0 u} \qquad \underbrace{\nabla\cdot v}_{d_0^* v}$$

is inf-sup stable if $\boxed{d_K W_K \subseteq W_{K+1}}$

So we want to build families of FEM

deRham complex



That will let us learn flux/circulation/potential balances

Construction of space (known as Whitney form construction)

Pick W_0 as our usual tent functions, $u \in W_0 \rightarrow u = \sum_i \hat{u}_i \psi_i^0(x)$

Take $\psi_i^0(x) = \phi_i(x)$

Define basis for W_1

$$\psi^1_{ij}(x) = \phi_i \nabla \phi_j - \phi_j \nabla \phi_i$$

$$u \in W_1 \rightarrow \quad u = \sum_{i,j} \hat{u}_{ij} \psi^1_{ij}(x)$$

Is $\nabla W_0 \subseteq W_1$?

Note $\sum_i' \phi_i = 1$

$\Rightarrow \sum_j' \nabla\phi_i = 0$

Pf Let $u \in W_0$

$$\rightarrow \nabla u = \sum_i' \hat{u}_i \nabla\phi_i$$
$$= \sum_i' \hat{u}_i \left[\nabla\phi_i \left(\sum_j \phi_j\right) - \left(\sum_j \nabla\phi_j\right)\phi_i\right]$$
$$= \sum_{i,j}' \hat{u}_i \psi_{ij}^1$$

$$\nabla u = \sum_{i,j}' \widehat{\nabla u}_{ij} \psi_{ij}^1 \Rightarrow \nabla u \in W_1$$

Skipping details (for HW)

$$\psi_{ijk}^2 = \phi_k \nabla\phi_i \times \nabla\phi_j + \phi_i \nabla\phi_k \times \nabla\phi_j + \phi_j \nabla\phi_i \times \nabla\phi_k$$

Could show

$\nabla \times u, \ u \in W_1 \rightarrow \nabla \times u \in W_2$

For next Hackathon / HW.

$$\nabla \cdot F = S$$

$u \in W_0$

$F \in W_1$

$$F = -\nabla u + N[u]$$

$$(F, v) + (\nabla u, v) - (N[u], v) = 0$$

$$-(F, \nabla q) = (S, w)$$

If we let $N[u] = \sum_i \eta[\hat{u}]_i \psi_i^0$

Then we get

$$\underbrace{\begin{pmatrix} M & G \\ G^T & 0 \end{pmatrix} \begin{pmatrix} \hat{F} \\ \hat{u} \end{pmatrix}}_{\substack{\text{inf-sup stable} \\ \text{(solvable)} \\ \text{diffusion}}} + \underbrace{\begin{pmatrix} M\,\eta[\hat{u}] \\ 0 \end{pmatrix}}_{\text{non linearity}} = \begin{pmatrix} 0 \\ S \end{pmatrix}$$

Think back to FDM! For a Lipschitz nonlinearity

$$Au + N[u] = b$$

is solvable!